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Advances in Mathematics 218 (2008) 216–265

ADVANCES IN Mathematics

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# Bispectral and $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ dualities, discrete versus differential

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Received 6 December 2006; accepted 29 November 2007

Available online 28 January 2008

Communicated by Michael J. Hopkins

**Abstract**

We define integral transforms which establish a correspondence between spaces of quasi-exponentials and spaces of quasi-polynomials under certain conditions. As a corollary of the properties of our transforms we obtain a correspondence between solutions to the Bethe ansatz equations of two $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ dual quantum integrable models: one is the special trigonometric Gaudin model and the other is the special XXX model.

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*Keywords:* Bethe ansatz; Critical point; Integral transform

## 1. Introduction

Let $V = \langle x^{\lambda_i} p_{ij}(x),\ i = 1, \dots, n,\ j = 1, \dots, N_i \rangle$ be a space of quasi-polynomials in $x$ of dimension $N = N_1 + \cdots + N_n$. The regularized fundamental differential operator of $V$ is the polynomial differential operator $\sum_{i=0}^{N} A_{N-i}(x)(x\partial_x)^i$ annihilating $V$ and such that its leading coefficient $A_0$ is a monic polynomial of the minimal possible degree.

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1 Supported in part by RFFI grant 05-01-00922.
2 Supported in part by NSF grant DMS-0244579.

0001-8708/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.aim.2007.11.022

Let $U = \langle z_a^u q_{ab}(u),\ a = 1, \dots, m,\ b = 1, \dots, M_a \rangle$ be a space of quasi-exponentials in $u$ of dimension $M = M_1 + \cdots + M_m$. The regularized fundamental difference operator of $U$ is the polynomial difference operator $\sum_{i=0}^{M} B_{M-i}(u)(\tau_u)^i$ annihilating $U$ and such that its leading coefficient $B_0$ is a monic polynomial of the minimal possible degree. Here $(\tau_u f)(u) = f(u+1)$.

We introduce the notion of a nondegenerate space of quasi-polynomials in Section 2 and the notion of a nondegenerate space of quasi-exponentials in Section 3.

Having a nondegenerate space $V$ of quasi-polynomials with the regularized fundamental differential operator $\sum_{i=0}^{N} A_{N-i}(x)(x\partial_x)^i$, we construct a nondegenerate space of quasi-exponentials $U = \langle z_a^u q_{ab}(u) \rangle$ whose regularized fundamental difference operator is the difference operator $\sum_{i=0}^{N} u^i A_{N-i}(\tau_u)$. The space $U$ is constructed from $V$ by a suitable integral transform.

Having a nondegenerate space $U = \langle z_a^u q_{ab}(u) \rangle$ of quasi-exponentials with the regularized fundamental difference operator $\sum_{i=0}^{M} B_{M-i}(u)(\tau_u)^i$, we construct a nondegenerate space of quasi-polynomials $V = \langle x^{\lambda_i} p_{ij}(x) \rangle$ whose regularized fundamental differential operator is the differential operator $\sum_{i=0}^{M} x^i B_{M-i}(x\partial_x)$. The space $V$ is constructed from $U$ by a suitable integral transform.

Our integral transforms are analogs of the bispectral involution on the space of rational solutions to the KP hierarchy [24].

As a corollary of the properties of our integral transforms we obtain a correspondence between solutions to the Bethe ansatz equations of two $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ dual quantum integrable models: one is the special trigonometric Gaudin model and the other is the special XXX model.

**Example.** Let $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{m} = (m_1, m_2)$ be two vectors of nonnegative integers such that $n_1 + n_2 = m_1 + m_2$. Let $d$ be the number of integers $i$ such that $\max(0, n_2 - m_1) \leqslant i \leqslant \min(m_2, n_2)$. Let $\boldsymbol{z} = (z_1, z_2), \boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ be two points with distinct coordinates.

Consider two systems of algebraic equations. The first system is the system of equations

$$\frac{\lambda_1 - \lambda_2 - 1}{t_i} + \sum_{a=1}^{2} \frac{m_a}{t_i - z_a} - \sum_{j=1, j\neq i}^{n_2} \frac{2}{t_i - t_j} = 0, \quad i = 1, \dots, n_2, \tag{1.1}$$

with respect to the unknown numbers $t_1, \dots, t_{n_2}$. The system is symmetric with respect to the group $\Sigma_{n_2}$ of permutations of $t_1, \dots, t_{n_2}$. One can show that for generic $\boldsymbol{z}, \boldsymbol{\lambda}$ the number of $\Sigma_{n_2}$-orbits of solutions of system (1.1) is equal to $d$. This system is called the system of the Gaudin Bethe ansatz equations, see Section 10.5.

The second systems of equations is the system

$$\prod_{i=1}^{2} \frac{s_a - \lambda_i - 1}{s_a - \lambda_i - 1 - n_i} \prod_{b, b\neq a}^{m_2} \frac{s_a - s_b - 1}{s_b - s_b + 1} = \frac{z_2}{z_1}, \quad a = 1, \dots, m_2, \tag{1.2}$$

with respect to the unknown numbers $s_1, \dots, s_{m_2}$. The system is symmetric with respect to the group $\Sigma_{m_2}$ of permutations of $s_1, \dots, s_{m_2}$. One can show that for generic $\boldsymbol{z}, \boldsymbol{\lambda}$ the number of $\Sigma_{m_2}$-orbits of solutions of system (1.2) with distinct $s_1, \dots, s_{m_2}$ is equal to $d$. This system is called the system of the XXX Bethe ansatz equations, see Section 10.5.

**Problem.** Establish a correspondence between orbits of solutions to systems (1.1) and (1.2).

We give two solutions to this problem.

*The first solution.* To system (1.1), we assign the vector space $(L_{m_1} \otimes L_{m_2})[n_1, n_2]$ and four commuting linear operators acting on this space. Here $(L_{m_1} \otimes L_{m_2})[n_1, n_2]$ denotes the weight subspace of weight $[n_1, n_2]$ of the tensor product of irreducible $\mathfrak{gl}_2$-modules with highest weights $(m_1, 0)$ and $(m_2, 0)$, respectively. The space $(L_{m_1} \otimes L_{m_2})[n_1, n_2]$ is of dimension $d$. The linear operators are denoted by

$$\begin{aligned} &H_1^{\mathfrak{G}}(\lambda_1 + n_1, \lambda_2 + n_2, z), \qquad &&H_2^{\mathfrak{G}}(\lambda_1 + n_1, \lambda_2 + n_2, z),\\ &G_1^{\mathfrak{G}}(\lambda_1 + n_1, \lambda_2 + n_2, z), \qquad &&G_2^{\mathfrak{G}}(\lambda_1 + n_1, \lambda_2 + n_2, z)\end{aligned}$$

and called the Gaudin KZ and dynamical Hamiltonians. To each orbit of solutions of system (1.1), the Bethe ansatz method assigns an eigenvector of the commuting Hamiltonians. The constructed Bethe vectors form a basis of this vector space.

To system (1.2), we assign the vector space $(L_{n_1} \otimes L_{n_2})[m_1, m_2]$ of the same dimension $d$ and four commuting linear operators acting on this space. The linear operators are denoted by

$$\begin{aligned} &H_1^{\mathfrak{X}}(z, \lambda_1 + n_1, \lambda_2 + n_2), \qquad &&H_2^{\mathfrak{X}}(z, \lambda_1 + n_1, \lambda_2 + n_2),\\ &G_1^{\mathfrak{X}}(z, \lambda_1 + n_1, \lambda_2 + n_2), \qquad &&G_2^{\mathfrak{X}}(z, \lambda_1 + n_1, \lambda_2 + n_2)\end{aligned}$$

and called the XXX KZ and dynamical Hamiltonians. To each orbit of solutions of system (1.2), the Bethe ansatz method assigns an eigenvector of the commuting Hamiltonians. The constructed Bethe vectors form a basis of this vector space.

There is a natural isomorphism of the vector spaces $(L_{m_1} \otimes L_{m_2})[n_1, n_2]$ and $(L_{n_1} \otimes L_{n_2})[m_1, m_2]$, which identifies the Hamiltonians:

$$\begin{aligned} H_a^{\mathfrak{G}}(\lambda_1 + n_1, \lambda_2 + n_2, z) &= G_a^{\mathfrak{X}}(z, \lambda_1 + n_1, \lambda_2 + n_2),\\ G_i^{\mathfrak{G}}(\lambda_1 + n_1, \lambda_2 + n_2, z) &= H_i^{\mathfrak{X}}(z, \lambda_1 + n_1, \lambda_2 + n_2),\end{aligned}$$

for $i, a = 1, 2$. This isomorphism is called the $(\mathfrak{gl}_2, \mathfrak{gl}_2)$-duality, see [22]. Under the duality isomorphism the Bethe vectors are identified and this identification gives a correspondence between the orbits of solutions of systems (1.1) and (1.2).

*The second solution.* To the orbit of a solution $(t_1, \dots, t_{n_2})$ of system (1.1), we assign the polynomial $p_2(x) = \prod_{i=1}^{n_2}(x - t_i)$ and the differential operator

$$D = x^2(x - z_1)(x - z_2)\left(\partial_x - \ln'\left(\frac{x^{\lambda_1 - 1}\prod_{a=1}^{2}(x - z_a)^{m_a}}{p_2}\right)\right)\left(\partial_x - \ln'\left(x^{\lambda_2} p_2\right)\right),$$

where $\partial_x = d/dx$ and for any function $f$, $\ln' f$ denotes the logarithmic derivative $f'/f$. Clearly the differential equation $Df(x) = 0$ has a solution $x^{\lambda_2} p_2(x)$.

We show that $D$ can be written in the form

$$D = A_0(x)(x\partial_x)^2 + A_1(x)x\partial_x + A_2(x)$$

where $A_0, A_1, A_2$ are polynomials in $x$ of degree not greater than two. Then we consider the second-order difference equation

$$\big(u^2 A_0(\tau_u) + u A_1(\tau_u) + A_2(\tau_u)\big) g(u) = 0$$

for the unknown function $g(u)$. It turns out that this difference equation has a solution of the form $z_2^u q_2(u)$, where $q_2(u) = \prod_{a=1}^{m_2}(u - s_a)$, and the roots $s_1, \dots, s_{m_2}$ form a solution of system (1.2). This construction gives a second correspondence between the orbits of solutions of systems (1.1) and (1.2).

We show that the two described constructions (the first, based on the $(\mathfrak{gl}_2, \mathfrak{gl}_2)$-duality, and the second, which uses the differential and difference operators) give the same correspondence between solutions of systems (1.1) and (1.2).

This paper is a development of results of the paper [10], in which we presented an integral transform giving an involution on the space of quasi-exponentials, the involution which corresponds to the bispectral involution of G. Wilson in [24].

The paper has the following structure. In Section 2, we discuss spaces of quasi-polynomials and their fundamental differential operators. In Section 3, we discuss spaces of quasi-exponentials and their fundamental difference operators. In Section 4, we define integral transforms establishing a duality between spaces of quasi-polynomials and quasi-exponentials. In Section 5, we introduce rigged spaces of quasi-polynomials and quasi-exponentials. We introduce rigged integral transforms relating the rigged spaces. In Section 6, we discuss relations between rigged spaces and solutions of the Bethe ansatz equations. The rigged spaces of quasi-polynomials correspond to solutions of the Bethe ansatz equations in the (trigonometric) Gaudin model. The rigged spaces of quasi-exponentials correspond to solutions of the Bethe ansatz equations in the XXX model. The number of solutions of the Bethe ansatz equations is discussed in Section 7. We describe the Gaudin and XXX models in Section 8. The $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality between the (trigonometric) Gaudin and XXX models is defined in Section 9. In Section 9, we formulate a conjecture about the bispectral correspondence of Bethe vectors under the $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality. In Section 10, we prove the conjecture for $N = M = 2$. In Section 11, we discuss the relation of our integral transforms with the bispectral correspondence of suitable Baker–Akhiezer functions for differential and difference operators.

## 2. Spaces of quasi-polynomials

### 2.1. Definition

Let $p \in \mathbb{C}[x]$ be a polynomial, $\lambda$ a complex number. The function $x^\lambda p(x)$ is called *a quasi-polynomial in $x$*. The quasi-polynomial is a multi-valued function. Different local univalued branches of the function differ by a nonzero constant factor.

Let $N_1, \dots, N_n$ be natural numbers. Set $N = N_1 + \dots + N_n$. For $i = 1, \dots, n$, let $0 < n_{i1} < \dots < n_{iN_i}$ be a sequence of positive integers. For $i = 1, \dots, n$, $j = 1, \dots, N_i$, let $p_{ij} \in \mathbb{C}[x]$ be a polynomial of degree $n_{ij}$.

Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be distinct numbers such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$.

Denote by $V$ the complex vector space spanned by functions $x^{\lambda_i} p_{ij}(x)$, $i = 1, \dots, n$, $j = 1, \dots, N_i$. The dimension of $V$ is $N$.

The space $V$ is called *the space of quasi-polynomials.*

We say that the space $V$ is *nondegenerate* if for any $i = 1, \dots, n$, and any $j = 1, \dots, N_i$,

- there exists a linear combination of polynomials $p_{i1}, p_{i2}, \dots, p_{iN_i}$ which has a root of multiplicity $j-1$ at $x=0$,
- the space $V$ does not contain the function $x^{\lambda_i+n_{ij}}$.

### 2.2. Exponents

Let $V$ be the space of quasi-polynomials.

For $z \in \mathbb{C}^*$, define *the sequence of exponents of $V$ at $z$* as the unique sequence of integers, $\boldsymbol{e} = \{e_1 < \dots < e_N\}$, with the property: for $i = 1, \dots, N$, there exists $f \in V$ such that $f$ has a root at $x = z$ of multiplicity $e_i$.

We say that $z \in \mathbb{C}^*$ is a singular point of $V$ if the set of exponents of $V$ at $z$ differs from the set $\{0, \dots, N-1\}$. The space $V$ has at most finitely many singular points.

Let $(z_1, \dots, z_m)$ be the subset of $\mathbb{C}^*$ of all singular points of $V$. For $a = 1, \dots, m$, let

$$\{0 < \dots < N - M_a - 1 < N - M_a + m_{a1} < \dots < N - M_a + m_{aM_a}\}$$

be the exponents of $V$ at $z_a$. Here $0 < m_{a1} < \dots < m_{aM_a}$ and $M_a$ is an integer such that $1 \leqslant M_a \leqslant N$. Set $M = M_1 + \dots + M_m$.

We say that $V$ is *a space of quasi-polynomials with data*

$$\mathfrak{D}_V = \{n, N_i, n_{ij}, \lambda_i, m, M_a, m_{ab}, z_a\}$$

where $i = 1, \dots, n$, $j = 1, \dots, N_i$, $a = 1, \dots, m$, $b = 1, \dots, M_a$.

### 2.3. Fundamental differential operator

For functions $f_1, \dots, f_i$ of one variable, denote by $\mathrm{Wr}(f_1, \dots, f_i)$ their Wronskian, that is, the determinant of the $i \times i$-matrix whose $j$th row is $f_j, f_j^{(1)}, \dots, f_j^{(i-1)}$.

Define the Wronskian of $V$, denoted by $\mathrm{Wr}_V$, as the Wronskian of a basis of $V$. The Wronskian of $V$ is determined up to multiplication by a nonzero number.

**Lemma 2.1.** *Let $V$ be a nondegenerate space of quasi-polynomials, then*

$$\sum_{a=1}^{m}\sum_{b=1}^{M_a}(m_{ab}+1-b) = \sum_{i=1}^{n}\sum_{j=1}^{N_i}(n_{ij}+1-j).$$

The lemma is proved by analyzing the order of zeroes of the Wronskian of $V$ and its asymptotics at infinity.

*The monic fundamental differential operator of $V$* is the unique monic linear differential operator of order $N$ annihilating $V$. It is denoted by $\bar{D}_V$. We have

$$\bar{D}_V = \partial_x^N + \bar{A}_1\partial_x^{N-1} + \dots + \bar{A}_N, \quad \bar{A}_i = (-1)^i \frac{\mathrm{Wr}_{V,i}}{\mathrm{Wr}_V},$$

where $\partial_x = d/dx$, $\mathrm{Wr}_V$ is the Wronskian of a basis $\{f_1, \dots, f_N\}$ of $V$, $\mathrm{Wr}_{V,i}$ is the determinant of the $N \times N$-matrix whose $j$th row is $f_j, f_j^{(1)}, \dots, f_j^{(N-i-1)}, f_j^{(N-i+1)}, \dots, f_j^{(N)}$.

For any $j = 1, \dots, N$, the order of the pole of $\bar{A}_j$ at $x = z_a$, $a = 1, \dots, m$, does not exceed $j$.

**Lemma 2.2.** *Let $V$ be a nondegenerate space of quasi-polynomials, then for $a = 1, \dots, m$, the order of the pole of $\bar{A}_{M_a}$ at $x = z_a$ is $M_a$ and the order of the pole of $\bar{A}_i$ is not greater than $M_a$ for $i > \bar{M}_a$.*

The proof follows from counting orders of determinants $W_{V,i}$.

The polynomial differential operator

$$D_V = x^N \prod_{a=1}^{m} (x - z_a)^{M_a} \bar{D}_V$$

is called *the regularized fundamental differential operator of $V$*.

It is easy to see that the regularized fundamental differential operator $D_V$ of $V$ has the form $\tilde{A}_0(x)x^N \partial_x^N + \tilde{A}_1(x)x^{N-1}\partial_x^{N-1} + \dots + \tilde{A}_N(x)$ where $\tilde{A}_i(x)$ are polynomials. Using the formula $x^i \partial_x^i = x\partial_x(x\partial_x - 1)\cdots(x\partial_x - i + 1)$ we can present the regularized fundamental differential operator in the form

$$D_V = A_0(x)(x\partial_x)^N + A_1(x)(x\partial_x)^{N-1} + A_2(x)(x\partial_x)^{N-2} + \dots + A_N(x)$$

with polynomial coefficients $A_i(x)$, $i = 0, \dots, N$.

**Lemma 2.3.** *Let $V$ be a nondegenerate space of quasi-polynomials, then*

- *We have $A_0(x) = \prod_{a=1}^m (x - z_a)^{M_a}$.*
- *The coefficients $A_0(x), \dots, A_N(x)$ are polynomials in $x$ of degree not greater than $M$. These polynomials have no common factor of positive degree.*
- *Write $D_V = x^M B_0(x\partial_x) + x^{M-1}B_1(x\partial_x) + \dots + xB_{M-1}(x\partial_x) + B_M(x\partial_x)$ where $B_0(x\partial_x), \dots, B_M(x\partial_x)$ are polynomials in $x\partial_x$ with constant coefficients. Then*

$$B_0(x\partial_x) = \prod_{i=1}^{n} \prod_{j=1}^{N_i} (x\partial_x - \lambda_i - n_{ij}), \quad B_M(x\partial_x) = \prod_{a=1}^{m} (-z_a)^{M_a} \prod_{i=1}^{n} \prod_{j=1}^{N_i} (x\partial_x - \lambda_i - j + 1).$$

- *The polynomials $B_0, \dots, B_M$ have no common factor of positive degree.*

The proof is straightforward.

*2.4. Conjugate space*

Let $V$ be a space of quasi-polynomials as in Section 2.1.

The complex vector space spanned by all functions of the form $\mathrm{Wr}(f_1, \dots, f_{N-1})/\mathrm{Wr}_V$ with $f_i \in V$ has dimension $N$. It is denoted by $V^\star$ and called *conjugate to $V$*.

The complex vector space spanned by all functions of the form $f(x)x^{-N} \prod_{a=1}^m (x - z_a)^{-M_a}$ with $f \in V^\star$ is denoted by $V^\dagger$ and called *regularized conjugate to $V$*.

**Lemma 2.4.** *Let $V$ be a nondegenerate space of quasi-polynomials, then*

(i) *For $a = 1, \dots, m$, if $\boldsymbol{e}$ are exponents of $V$ at $z_a$, then*

$$\boldsymbol{e}^\star = \{-e_N - 1 + N < -e_{N-1} - 1 + N < \cdots < -e_1 - 1 + N\}$$

*are exponents of $V^\star$ at $z_a$ and*

$$\boldsymbol{e}^\dagger = \{-e_N - 1 + N - M_a < -e_{N-1} - 1 + N - M_a < \cdots < -e_1 - 1 + N - M_a\}$$

*are exponents of $V^\dagger$ at $z_a$.*

(ii) *For any $i = 1, \dots, n$, $j = 1, \dots, N_i$, there exists $f \in V^\dagger$ such that the function $x^{\lambda_i + j} f$ has a nonzero limit as $x \to 0$.*

(iii) *For any $i = 1, \dots, n$, $j = 1, \dots, N_i$, there exists $f \in V^\dagger$ such that the function $x^{\lambda_i + n_{ij} + 1 - N_i} f$ has a nonzero limit as $x \to \infty$.*

The proof is straightforward.

Let $D = \sum_i A_i(x)\partial_x^i$ be a differential operator with meromorphic coefficients. The operator $D^* = \sum_i (-\partial_x)^i A_i(x)$ is called *formal conjugate to $D$*.

**Lemma 2.5.** *Let $V$ be a nondegenerate space of quasi-polynomials. Let $\bar{D}_V$ and $D_V$ be the monic and regularized fundamental differential operators of $V$, respectively. Then $(\bar{D}_V)^*$ annihilates $V^\star$ and $(D_V)^*$ annihilates $V^\dagger$.*

The proof is straightforward.

## 3. Spaces of quasi-exponentials

### *3.1. Definition*

Define the operator $\tau_u$ acting on functions of $u$ as $(\tau_u f)(u) = f(u+1)$.

Let $z$ be a nonzero complex number with fixed argument. Set $z^u = e^{u \ln z}$. We have $\tau_u z^u = z^u z$.

Let $q \in \mathbb{C}[u]$ be a polynomial. The function $z^u q(u)$ is called *a quasi-exponential in $u$*.

Let $M_1, \dots, M_m$ be natural numbers. Set $M = M_1 + \cdots + M_m$. For $a = 1, \dots, m$, let $0 < m_{a1} < \cdots < m_{aM_a}$ be a sequence of positive integers. For $a = 1, \dots, m$, $b = 1, \dots, M_a$, let $q_{ab} \in \mathbb{C}[u]$ be a polynomial of degree $m_{ab}$.

Let $z_1, \dots, z_m$ be distinct nonzero complex numbers with fixed arguments.

Denote by $U$ the complex vector space spanned by functions $z_a^u q_{ab}(u)$, $a = 1, \dots, m$, $b = 1, \dots, M_a$. The dimension of $U$ is $M$.

The space $U$ is called *the space of quasi-exponentials.*

For functions $f_1, \dots, f_i$ of $u$, denote by $\mathrm{Wr}^{(\mathrm{d})}(f_1, \dots, f_i)$ their discrete Wronskian which is the determinant of the $i \times i$-matrix whose $j$th row is $f_j(u), f_j(u+1), \dots, f_j(u+i-1)$.

Define *the discrete Wronskian of $U$*, denoted by $\mathrm{Wr}_U^{(\mathrm{d})}$, as the discrete Wronskian of a basis of $U$. The discrete Wronskian of $U$ is determined up to multiplication by a nonzero constant.

**Lemma 3.1.** *We have*

$$\mathrm{Wr}_U^{(\mathrm{d})}(u) = S(u) \prod_{a=1}^{m} z_a^{M_a u},$$

*where $S(u)$ is a polynomial of degree $\sum_{a=1}^{m} \sum_{b=1}^{M_a} (m_{ab} + 1 - b)$.*

The proof is straightforward.

*3.2. The frame of a space of quasi-exponentials*

Let $U$ be a space of quasi-exponentials like in Section 3.1. Let $v_1, \dots, v_M$ be the quasi-exponentials generating $U$.

For $i = 1, \dots, M$, let $S_i \in \mathbb{C}[u]$ be the monic polynomial of the greatest degree such that the function $\mathrm{Wr}^{(\mathrm{d})}(v_{j_1}, v_{j_2}, \dots, v_{j_i})/S_i$ is regular for any $j_1, j_2, \dots, j_i \in \{1, \dots, M\}$.

In particular, for the discrete Wronskian of $U$ we have

$$\mathrm{Wr}_U^{(\mathrm{d})}(u) = \mathrm{const}\, S_M(u) \prod_{a=1}^{m} z_a^{M_a u}$$

with a nonzero constant.

**Lemma 3.2.** *There exists a unique sequence of monic polynomials $P_1(u), \dots, P_M(u)$ such that*

$$S_i(u) = \prod_{k=1}^{i} \prod_{j=1}^{i-k+1} P_k(u + j - 1)$$

*for $i = 1, \dots, M$.*

This lemma is an analog of Lemma 4.12 in [14].

The monic polynomials $P_1(u), \dots, P_M(u)$ are called *the frame* of $U$.

**Proof.** We construct $P_i$ by induction on $i$. For $i = 1$, we set $P_1 = S_1$. Suppose the lemma is proved for all $i = 1, \dots, i_0 - 1$. Then we set

$$R(u) = \prod_{i=1}^{i_0-1} \prod_{j=1}^{i_0-i+1} P_i(u + j - 1), \qquad P_{i_0}(u) = S_{i_0}(u)/R(u).$$

We have to show that $P_{i_0}$ is a polynomial. In other words, we have to show that the discrete Wronskian of any $i_0$-dimensional subspace $\langle v_{j_1}, \dots, v_{j_{i_0}} \rangle$ is divisible by $R(u)$.

Consider the Grassmannian $\mathrm{Gr}(i_0 - 2, U)$ of $(i_0 - 2)$-dimensional spaces in $U$. For any $z \in \mathbb{C}$ the set of points in $\mathrm{Gr}(i_0 - 2, U)$ such that the corresponding discrete Wronskian divided by $S_{i_0-2}$ does not vanish at $z$, is an open set. Therefore, we have an open set of points in $\mathrm{Gr}(i_0 - 2, U)$ such that the corresponding discrete Wronskian divided by $S_{i_0-2}$ does not vanish at roots of $R(u - 1)$. We call such subspaces acceptable.

Therefore, we have an open set of points in $\mathrm{Gr}(i_0, U)$ such that the corresponding $i_0$-dimensional space contains an acceptable $i_0 - 2$ dimensional subspace. Let $w_1, \dots, w_{i_0} \in U$ be such that $w_1, \dots, w_{i_0-2}$ span an acceptable space. It is enough to show that $\mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0})$ is divisible by $R(u)$.

Using discrete Wronskian identities of [14], we have for suitable holomorphic functions $f_1, f_2, g$:

$$\begin{aligned}\mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0}) &= \frac{\mathrm{Wr}^{(\mathrm{d})}(\mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0-1}), \mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0-2}, w_{i_0}))}{\mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0-2})(u+1)} \\ &= \frac{\mathrm{Wr}^{(\mathrm{d})}(S_{i_0-1} f_1, S_{i_0-1} f_2)}{S_{i_0-2}(u+1) g(u+1)} = \frac{S_{i_0-1}(u) S_{i_0-1}(u+1)}{S_{i_0-2}(u+1)} \frac{\mathrm{Wr}^{(\mathrm{d})}(f_1, f_2)}{g(u+1)} \\ &= R(u) \frac{\mathrm{Wr}^{(\mathrm{d})}(f_1, f_2)}{g(u+1)}.\end{aligned}$$

Since the space spanned by $w_1, \dots, w_{i_0-2} \in U$ is acceptable, the functions $g(u+1) = \mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0-2})(u+1)/S_{i_0-2}(u+1)$ and $R(u)$ do not have common zeros. Therefore, the discrete Wronskian $\mathrm{Wr}^{(\mathrm{d})}(w_1, \dots, w_{i_0})$ is divisible by $R(u)$. $\square$

### 3.3. *Discrete exponents*

For $\lambda \in \mathbb{C}$, there exists an increasing sequence of nonnegative integers $\{c_1 < \dots < c_M\}$ and a basis $\{f_1, \dots, f_M\}$ of $U$ such that for $i = 1, \dots, M$, we have $f_i(\lambda + j) = 0$ for $j = 0, \dots, c_i - 1$ and $f_i(\lambda + c_i) \neq 0$. This sequence of integers is defined uniquely and will be called *the sequence of discrete exponents of $U$ at $\lambda$*. We say that the basis $\{f_1, \dots, f_M\}$ agrees with exponents at $\lambda$.

We say that $\lambda$ is *a singular point of $U$* if the discrete exponents at $\lambda$ differ from the sequence $\{0 < 1 < \dots < M-1\}$.

For $i = 1, \dots, M$, introduce *the local frame-type polynomials*

$$Q_i = \prod_{j=c_{i-1}-i+2}^{c_i - i} (u - \lambda - j)$$

where $c_0 = -1$. Notice that

- roots of each $Q_i$ are simple,
- $\deg Q_i = c_i - c_{i-1} - 1$,
- sets of roots of different polynomials do not intersect,
- the union of roots of all $Q_i$ is the sequence $\lambda, \lambda + 1, \dots, \lambda + c_M - M$.

**Theorem 3.3.** *Let $c_1 < \dots < c_M$ be the discrete exponents of $U$ at $\lambda$. Then*

(i) *The discrete Wronskian of $U$ is divisible by*

$$\prod_{s=1}^{M} \prod_{j=s-M}^{c_s - M} (u - \lambda - j) = \prod_{i=1}^{M} \prod_{j=1}^{M-i+1} Q_i(u + j - 1).$$

*In particular, the total degree of this divisor is* $\sum_{i=1}^{M}(c_i - i + 1)$.

(ii) *If* $S_1, \dots, S_M$ *are the polynomials from Lemma* 3.2*, then for any* $k$*, the polynomial* $S_k$ *is divisible by*

$$\prod_{i=0}^{k} \prod_{j=1}^{k-i+1} Q_i(u + j - 1).$$

**Corollary 3.4.** *Assume that the exponents of* $U$ *at* $\lambda$ *have the form*

$$0 < \cdots < M - L - 1 < M - L + l_1 < M - L + l_2 < \cdots < M - L + l_L$$

*for a suitable* $L$, $1 \leqslant L \leqslant M$*, and* $0 < l_1 < \cdots < l_L$*. Then the local frame-type polynomials have the form* $Q_i = 1$ *for* $i = 1, \dots, M - L$,

$$Q_{M-L+1} = \prod_{j=0}^{l_1-1} (u - \lambda - j), \qquad Q_{M-L+i} = \prod_{j=l_{i-1}-i+2}^{l_i - i} (u - \lambda - j)$$

*for* $i = 2, \dots, L$*, and the discrete Wronskian of* $U$ *is divisible by*

$$\prod_{k=1}^{L} \prod_{j=k-L}^{l_k - L} (u - \lambda - j).$$

*In particular, the total degree of this divisor is* $\sum_{k=1}^{L}(l_k - k + 1)$.

### 3.4. Proof of Theorem 3.3

We shall prove part (i). Part (ii) is proved analogously.

Let $\{f_1, \dots, f_M\}$ be a basis in $U$ that agrees with exponents at $\lambda$. Consider the matrix-valued function $F(u) = [f_j(u + k - 1)]_{j,k=1,\dots,M}$.

**Lemma 3.5.** *For* $t \in \mathbb{C}$*, if the corank of* $F(t)$ *is* $r$*, then the discrete Wronskian* $\mathrm{Wr}^{(\mathrm{d})}(f_1, \dots, f_M)$ *is divisible by* $(u - t)^r$.

The proof is straightforward.

For $j \in \mathbb{Z}$, set $d_j = \#\{s \leqslant M \mid j + M \leqslant c_s\}$. For $j \geqslant 0$, set $r_j = d_j$, and for $j < 0$, set $r_j = \max(0, d_j + j)$.

It is easy to see that for $j < 0$, the number $r_j$ can be also defined as $\#\{s \leqslant M \mid s \leqslant j + M \leqslant c_s\}$.

**Lemma 3.6.** *For* $j \in \mathbb{Z}$*, the corank of* $F(\lambda + j)$ *is not less than* $r_j$.

**Proof.** If $j \geqslant 0$, then $F(\lambda + j)$ has $d_j$ zero rows produced by the functions $f_{M-d_j+1}, \dots, f_M$. Hence, the corank of $F(\lambda + j)$ is at least $d_j = r_j$.

If $-M \leqslant j < 0$, then the rows produced by the functions $f_{M-d_j+1}, \dots, f_M$ have zeros everywhere except in the first $-j$ columns. Hence, the corank of $F(\lambda + j)$ is at least $d_j - (-j)$. □

By Lemma 3.6, the discrete Wronskian of $U$ is divisible by $\prod_{j=1-M}^{c_M-M}(u-\lambda-j)^{r_j}$. That can be written as $\prod_{s=1}^{M}\prod_{j=s-M}^{c_s-M}(u-\lambda-j)$. Theorem 3.3 is proved.

### 3.5. *Numerically nondegenerate space of quasi-exponentials*

Let $U$ be a space of quasi-exponentials like in Section 3.1.

Let $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ be such that $\lambda_i-\lambda_j\notin\mathbb{Z}$ for $i\neq j$. For $i=1,\dots,n$, let the exponents of $U$ at $\lambda_i$ have the form

$$\{0<\cdots<M-N_i-1<M-N_i+n_{i1}<M-N_i+n_{i2}<\cdots<M-N_i+n_{iN_i}\}$$

for a suitable $N_i$, $1\leqslant N_i\leqslant M$, and $0<n_{i1}<\cdots<n_{iN_i}$. Set

$$N=N_1+\cdots+N_n.$$

We say that the space $U$ is *a space of quasi-exponentials with data*

$$\mathfrak{D}=\{m,M_a,m_{ab},z_a,n,N_i,n_{ij},\lambda_i\}$$

where $a=1,\dots,m$, $b=1,\dots,M_a$, $i=1,\dots,n$, $j=1,\dots,N_i$.

We say that the space $U$ is *a numerically nondegenerate space of quasi-exponentials with respect to data* $\mathfrak{D}$ if

$$\sum_{a=1}^{m}\sum_{b=1}^{M_a}(m_{ab}+1-b)=\sum_{i=1}^{n}\sum_{j=1}^{N_i}(n_{ij}+1-j). \tag{3.1}$$

Clearly, $\lambda_1,\dots,\lambda_n$ are singular points of the space $U$, but not necessarily all of the singular points of $U$.

For $i=1,\dots,n$, let $Q_{i1},\dots,Q_{iM}$ be the local frame-type polynomials associated with the point $\lambda_i$. For $k=1,\dots,M$, define

$$Q_k(u)=\prod_{i=1}^{n}Q_{ik}(u). \tag{3.2}$$

**Lemma 3.7.** *If $U$ is a numerically nondegenerate space of quasi-exponentials, then the discrete Wronskian of $U$ is given by the formula*

$$\mathrm{Wr}_U^{(\mathrm{d})}=\prod_{a=1}^{m}z_a^{M_a u}\prod_{k=1}^{M}\prod_{j=1}^{M-k+1}Q_k(u+j-1).$$

*Moreover, if $S_1,\dots,S_M$ are the polynomials from Lemma* 3.2*, then for any $i$, the polynomial $S_i$ is divisible by*

$$\prod_{k=0}^{i}\prod_{j=1}^{i-k+1}Q_k(u+j-1).$$

The lemma follows from Lemmas 3.1, 3.2, and Theorem 3.3.

### 3.6. Fundamental difference operator

*The monic fundamental difference operator of a space of quasi-exponentials $U$* is the unique monic linear difference operator

$$\bar{D}_U = \tau_u^M + \bar{B}_1 \tau_u^{M-1} + \cdots + \bar{B}_{M-1}\tau_u + \bar{B}_M$$

of order $M$ annihilating $U$. Here

$$\bar{B}_i = (-1)^i \frac{\mathrm{Wr}^{(\mathrm{d})}_{U,i}}{\mathrm{Wr}^{(\mathrm{d})}_U},$$

where $\mathrm{Wr}^{(\mathrm{d})}_U$ is the discrete Wronskian of a basis $\{f_1, \ldots, f_M\}$ of $U$, $\mathrm{Wr}^{(\mathrm{d})}_{U,i}$ is the determinant of the $M \times M$-matrix whose $j$th row is

$$f_j(u),\ f_j(u+1), \ldots, f_j(u+N-i-1),\ f_j(u+N-i+1), \ldots, f_j(u+N).$$

Clearly, $\bar{B}_1, \ldots, \bar{B}_M$ are rational functions.

**Lemma 3.8.** *For any $i$, the function $\bar{B}_i$ has a limit as $u$ tends to infinity. Denote this limit by $\bar{B}_i(\infty)$. Then*

$$x^M + \bar{B}_1(\infty)x^{M-1} + \cdots + \bar{B}_M(\infty) = \prod_{a=1}^{M}(x - z_a)^{M_a}.$$

The proof is straightforward.

**Lemma 3.9.** *For any $i = 1, \ldots, M$, the function*

$$\bar{B}_i(u)\prod_{i=1}^{n}\prod_{j=1}^{N_i}(u - \lambda_i - n_{ij} + N_i)$$

*is a polynomial.*

**Proof.** Let $\lambda$ be one of the points of the set $\{\lambda_1, \ldots, \lambda_n\}$. For such a $\lambda$, in the proof of Theorem 3.3, we defined the numbers $d_j$ and $r_j$ for $j \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, define the new numbers $p_j$ as follows. Set $p_j = d_{j+1}$ for $j \geqslant 0$, and set $p_j = \max(0, j + d_{j+1})$ for $j < 0$.

By the construction, we have $p_j \leqslant r_j$.

The reasons in the proof of Theorem 3.3 show that the discrete Wronskian $\mathrm{Wr}^{(\mathrm{d})}_U$ is divisible by

$$X(u) = \prod_{j=1-M}^{c_M - M}(u - \lambda - j)^{r_j}.$$

Similar reasons show that for any $k$, the determinant $\mathrm{Wr}^{(\mathrm{d})}_{U,k}$ is divisible by

$$Y(u) = \prod_{j=1-M}^{c_M-M-1} (u-\lambda-j)^{p_j}.$$

As at the end of the proof of Theorem 3.3, we have

$$X(u) = \prod_{s=1}^{M} \prod_{j=s-M}^{c_s-M} (u-\lambda-j) = \prod_{\substack{s=1\\ c_s \geqslant s}}^{M} \prod_{j=s-M}^{c_s-M} (u-\lambda-j),$$

where in the second expression we excluded the empty products over $j$. Similarly,

$$Y(u) = \prod_{\substack{s=1\\ c_s \geqslant s+1}}^{M} \prod_{j=s-M}^{c_s-1-M} (u-\lambda-j) = \prod_{\substack{s=1\\ c_s \geqslant s}}^{M} \prod_{j=s-M}^{c_s-1-M} (u-\lambda-j),$$

where the first expression is analogous to the second expression for $X(u)$, and the second expression may contain certain empty products over $j$.

Using the second expressions for $X(u)$ and $Y(u)$, we get

$$X(u) = Y(u) \prod_{\substack{s=1\\ c_s \geqslant s}}^{M} (u-\lambda-c_s+M). \tag{3.3}$$

Now if $\lambda=\lambda_i$, we shall provide $X(u)$ and $Y(u)$ with index $i$. Calculating the product in (3.3) for $\lambda=\lambda_i$, we get

$$X_i(u) = Y_i(u) \prod_{j=1}^{N_i} (u-\lambda_i-n_{ij}+N_i).$$

Multiplying this formula over $i=1,\dots,n$, we conclude that for any $k$, the product

$$\mathrm{Wr}^{(\mathrm{d})}_{U,k}(u) \prod_{i=1}^{n} \prod_{j=1}^{N_i} (u-\lambda_i-n_{ij}+N_i)$$

is divisible by the discrete Wronskian $\mathrm{Wr}^{(\mathrm{d})}_{U}$. $\square$

Define *the regularized fundamental difference operator of the space $U$* as the linear difference operator

$$D_U = B_0(u)\tau_u^M + B_1(u)\tau_u^{M-1} + \cdots + B_M(u)$$

of order $M$ with polynomial coefficients, which annihilates $U$ and such that its leading coefficient $B_0$ is a monic polynomial of the minimal possible degree.

We have

$$\deg B_0 = \deg B_M \geqslant \deg B_i \quad \text{for } i = 1, \ldots, M-1,$$

by Lemma 3.8.

We say that the space $U$ with data $\{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$ is *a nondegenerate space of quasi-exponentials* if $U$ is numerically nondegenerate and

$$\deg B_0 = N = N_1 + \cdots + N_n.$$

By Lemma 3.9, if the space $U$ with data $\{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$ is nondegenerate, then

$$B_0 = \prod_{i=1}^{n} \prod_{j=1}^{N_i} (u - \lambda_i - n_{ij} + N_i).$$

**Example.** Let $U$ be the vector space spanned by the quasi-exponentials $u$ and $u(u-1)$. Then $M = 2$, $m = 1$, $z_1 = 1$, $m_{11} = 1$, $m_{12} = 2$. Let $\lambda_1 = 0$. The exponents of $U$ at $\lambda_1$ are 1 and 2. Then $n_{11} = 1$, $n_{12} = 2$, $N = N_1 = 2$. Equality (3.1) takes the form: $2 = 2$. We have

$$D_U = u(u+1)\tau_u^2 - 2u(u+2)\tau_u + (u+1)(u+2).$$

Hence $U$ with this data is nondegenerate.

**Example.** Let $U$ be the vector space spanned by the quasi-exponentials $u$ and $(-1)^u u$. Then $M = 2$, $m = 2$, $z_1 = 0$, $z_2 = -1$, $M_1 = 1$, $m_{11} = 1$, $M_2 = 1$, $m_{21} = 1$. Let $\lambda_1 = 0$. The exponents of $U$ at $\lambda_1$ are 1 and 2. Then $n_{11} = 1$, $n_{12} = 2$, $N = N_1 = 2$. Equality (3.1) takes the form: $2 = 2$. Hence $U$ with this data is numerically nondegenerate. We have

$$D_U = u\tau_u^2 + (u+2).$$

Hence $U$ with this data is degenerate.

**Example.** Let $U$ be the vector space spanned by the quasi-exponentials $u$ and $(-1)^u u$. Then $M = 2$, $m = 2$, $z_1 = 0$, $z_2 = -1$, $M_1 = 1$, $m_{11} = 1$, $M_2 = 1$, $m_{21} = 1$. Let $\lambda_1 = -1$. The exponents of $U$ at $\lambda_1$ are $0, 3$. Then $N_1 = 1$, $n_{11} = 2$, $N = 1$ and equality (3.1) is $2 = 2$. Therefore, $U$ with this data is numerically nondegenerate. With this data, we have

$$D_U = u\tau_u^2 + (u+2).$$

Hence, $U$ with this data is nondegenerate.

**Theorem 3.10.** *Assume that the space $U$ with data $\{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$ is a nondegenerate space of quasi-exponentials. Then*

(i) *We have*

$$B_M = \prod_{a=1}^{m} (-z_a)^{M_a} \prod_{i=1}^{n} \prod_{j=1}^{N_i} (u - \lambda_i + j).$$

(ii) *Write*

$$D_U = u^N A_0(\tau_u) + u^{N-1} A_1(\tau_u) + \cdots + A_N(\tau_u)$$

*where $A_i(\tau_u)$ is a polynomial in $\tau_u$ with constant coefficients. Then*

$$A_0(\tau_u) = \prod_{a=1}^{m} (\tau_u - z_a)^{M_a}.$$

(iii) *The polynomials $A_0, \dots, A_M$ have no common factors of positive degree.*

**Corollary 3.11.** *If the space $U$ is nondegenerate with respect to a data $\{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$, then the data is determined uniquely.*

The corollary follows from part (i) of the theorem.

**Proof of Theorem 3.10.** Part (ii) follows from Lemma 3.8. Part (iii) follows from the fact that $U$ does not contain exponential functions $z^u$.

Let $Q_1, \dots, Q_M$ be the polynomials introduced in (3.2). To prove part (i) it is enough to notice that

$$\frac{B_M(u)}{B_0(u)} = (-1)^M \frac{\mathrm{Wr}^{(\mathrm{d})}(u+1)}{\mathrm{Wr}^{(\mathrm{d})}(u)} = \prod_{k=1}^{M} \frac{Q_k(u+M+1-k)}{Q_k(u)} \prod_{a=1}^{m} z_a^{M_a}$$
$$= (-1)^M \prod_{i=1}^{n} \prod_{j=1}^{N_i} \frac{u - \lambda_i + j}{u - \lambda_i - n_{ij} + N_i} \prod_{a=1}^{m} z_a^{M_a}. \qquad \square$$

*3.7. Regularized conjugate space*

Let $U$ with data $\{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$ be a nondegenerate space of quasi-exponentials as in Section 3.6. Let $\mathrm{Wr}_U^{(\mathrm{d})}$ be the discrete Wronskian of $U$ and let $B_M(u)$ be the last coefficient of the regularized fundamental difference operator of $U$.

The complex vector space spanned by all functions of the form

$$\frac{\tau_u(\mathrm{Wr}^{(\mathrm{d})}(f_1, \dots, f_{M-1}))}{B_M(u)\mathrm{Wr}_U^{(\mathrm{d})}(u)} \tag{3.4}$$

with $f_i \in V$ has dimension $M$. This space is denoted by $U^{\ddagger}$ and called *regularized conjugate to $U$*.

**Lemma 3.12.** *For any $g \in U^{\ddagger}$, the function*

$$g(u)\prod_{i=1}^{n}\prod_{j=0}^{n_{iN_i}}(u-\lambda_i+N_i-j)$$

*is holomorphic in $\mathbb{C}$.*

**Proof.** Let $Q_1, \dots, Q_M$ be the polynomials introduced in (3.2). Let $g$ be a function in (3.4). Then $\tau_u(\mathrm{Wr}^{(\mathrm{d})}(f_1, \dots, f_{M-1}))$ is divisible by

$$\prod_{i=1}^{M-1} Q_1(u+i)\cdot\prod_{i=1}^{M-2} Q_2(u+i)\cdots Q_{M-1}(u+1)$$

while

$$\mathrm{Wr}_U^{(\mathrm{d})}(u)=\prod_{a=1}^{m} z_a^{M_a}\prod_{j=1}^{M} Q_1(u+j-1)\cdot\prod_{j=1}^{M-1} Q_2(u+j-1)\cdots Q_M(u).$$

Hence, the possible poles of $g$ come from the product $B_M(u)Q_1(u)\cdots Q_M(u)$ which remains in the denominator of $g$. But this product is exactly the product in Lemma 3.12. □

For every $i=1,\dots,n$ and $j=1,\dots,N_i$, fix a function $g_{ij}$ in $U$ which is equal to zero at $u=\lambda_i, \lambda_i+1, \dots, \lambda_i+M-N_i+n_{ij}-1$ and which is not equal to zero at $u=\lambda_i+M-N_i+n_{ij}$.

**Lemma 3.13.** *For given $i=1,\dots,n$, $j=0,\dots,N_i-1$, let $f_1,\dots,f_{M-1}$ be a collection of functions in $U$ containing the functions $g_{i,j+1}, g_{i,j+2}, \dots, g_{i,N_i}$ and let*

$$F(u)=\frac{\tau_u(\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1}))}{B_M(u)\mathrm{Wr}_U^{(\mathrm{d})}(u)}.$$

*Then for $j=0$, the function $F$ has no poles at*

$$u=\lambda_i-N_i, \lambda_i-N_i+1, \dots, \lambda_i-N_i+n_{iN_i}.$$

*For $j>0$, the function $F$ has no poles at*

$$u=\lambda_i-N_i+n_{ij}+1, \lambda_i-N_i+n_{ij}+2, \dots, \lambda_i-N_i+n_{iN_i}.$$

*Moreover, if $f_1,\dots,f_{M-1}$ is a generic collection of functions in $U$ containing the functions $g_{i,j+1}, g_{i,j+2}, \dots, g_{i,N_i}$, then the function $F$ has a nonzero residue at $u=\lambda_i-N_i+n_{ij}$.*

**Proof.** The first two statements of the lemma are proved in the same way as Lemma 3.12.

We shall prove that the residue of $F$ at $u=\lambda_i-N_i+n_{ij}$ is nonzero first assuming that $n_{ij}\geqslant N_i$. To prove that the residue is nonzero it is enough to show that

$$\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}}\mathrm{Wr}_U^{(\mathrm{d})}=1+\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}_U^{(\mathrm{d})}=N_i-j+1, \tag{3.5}$$

and

$$\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}_U^{(\mathrm{d})}=\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1}). \tag{3.6}$$

Here $\mathrm{ord}_{u=\lambda} f$ denotes the order of zero of the function $f$ at $u=\lambda$.

Equalities (3.5) follow from the numerical nondegeneracy of the space $U$ and the condition $n_{ij}\geqslant N_i$.

Since the collection $f_1,\dots,f_{M-1}$ is a generic collection containing the functions $g_{i,j+1},\dots,g_{i,N_i}$, the discrete Wronskian $\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1},g_{ij})$ is nonzero and proportional to $\mathrm{Wr}_U^{(\mathrm{d})}$. Expanding the determinant $\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1},g_{ij})$ with respect to the last row, we have

$$\begin{aligned}\mathrm{Wr}_U^{(\mathrm{d})}(u)=\mathrm{const}\big(&g_{ij}(u+M-1)\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1})(u)-\cdots\\&-(-1)^M g_{ij}(u)\tau_u\big(\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1})\big)(u)\big)\end{aligned} \tag{3.7}$$

where $\mathrm{const}\neq 0$.

The order of $\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1})$ at $u=\lambda_i-N_i+n_{ij}+1$ is not less than $N_i-j$. This follows from Theorem 3.3 applied to the functions $f_1,\dots,f_{M-1}$. A similar reason shows that the order at $u=\lambda_i-N_i+n_{ij}+1$ of all of the other $(M-1)\times(M-1)$ minors in the right-hand side of (3.7) is also not less than $N_i-j$.

By the construction, the function $g_{ij}$ is nonzero at $u=\lambda_i-N_i+n_{ij}$ and is zero at $u=\lambda_i-N_i+n_{ij}-l$ for $l=1,\dots,M-1$. Therefore, the only term in the right-hand side of equality (3.7) that can have order $N_i-j$ at $u=\lambda_i-N_i+n_{ij}+1$ is $g_{ij}(u+M-1)\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1})(u)$. Since $\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}_U^{(\mathrm{d})}=N_i-j$, this shows that the orders of $\mathrm{Wr}_U^{(\mathrm{d})}$ and $\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1})$ at $u=\lambda_i-N_i+n_{ij}$ are equal.

To prove that the residue of $F$ at $u=\lambda_i-N_i+n_{ij}$ is nonzero in the case $n_{ij}<N_i$, it is enough to show that

$$\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}}\mathrm{Wr}_U^{(\mathrm{d})}=\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}_U^{(\mathrm{d})}=n_{ij}-j+1,$$

and

$$\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}_U^{(\mathrm{d})}=\mathrm{ord}_{u=\lambda_i-N_i+n_{ij}+1}\mathrm{Wr}^{(\mathrm{d})}(f_1,\dots,f_{M-1}).$$

The proof is similar to the proof of equalities (3.5) and (3.6). □

**Theorem 3.14.** *If $D_U=B_0(u)\tau_u^M+B_1(u)\tau_u^{M-1}+\cdots+\tau_u B_{M-1}(u)+B_M(u)$ is the regularized fundamental difference operator of $U$, then the operator*

$$D_U^{\ddagger}=\tau_u^M B_M(u)+\tau_u^{M-1}B_{M-1}(u)+\cdots+\tau_u B_1(u)+B_0(u)$$

*annihilates $U^{\ddagger}$.*

**Proof.** Consider the scalar equation $D_U y=0$ with respect to an unknown function $y(u)$. For $i=1,\dots,M$, introduce $w_i=\tau_u^{i-1}y$ and present the equation as a system of first order equations

$$\tau_u w_M=-\frac{B_1(u)}{B_0(u)}w_M-\cdots-\frac{B_M(u)}{B_0(u)}w_1,\qquad \tau_u w_i=w_{i+1}$$

for $i = 1, \dots, M-1$. For the column vector $w = (w_1, \dots, w_M)$, the system can be presented as a matrix equation $\tau_u w = Cw$ with the $M \times M$-matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{B_M(u)}{B_0(u)} & -\frac{B_{M-1}(u)}{B_0(u)} & -\frac{B_{M-2}(u)}{B_0(u)} & \dots & -\frac{B_2(u)}{B_0(u)} & -\frac{B_1(u)}{B_0(u)} \end{pmatrix}.$$

Let $\Psi$ be a fundamental $M \times M$-matrix of solutions, $\tau_u \Psi = C\Psi$. Then $\tau_u \Psi^{-1} = \Psi^{-1} C^{-1}$ where

$$C^{-1} = \begin{pmatrix} -\frac{B_{M-1}(u)}{B_M(u)} & -\frac{B_{M-2}(u)}{B_M(u)} & -\frac{B_{M-3}(u)}{B_M(u)} & \dots & -\frac{B_1(u)}{B_M(u)} & -\frac{B_0(u)}{B_M(u)} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

For a row vector $(v_1, \dots, v_M)$ the equation $\tau_u v = vC^{-1}$ has the form:

$$\tau_u v_M = -\frac{B_0}{B_M} v_1, \quad \tau_u v_{M-1} = -\frac{B_1}{B_M} v_1 + v_M,$$

$$\tau_u v_{M-2} = -\frac{B_2}{B_M} v_1 + v_{M-1}, \quad \dots, \quad \tau_u v_1 = -\frac{B_{M-1}}{B_M} v_1 + v_2.$$

This system reduces to the scalar equation $D_U^\star v_1 = 0$ where

$$D_U^\star = \tau_u^M + \tau_u^{M-1} \frac{B_{M-1}(u)}{B_M(u)} + \dots + \tau_u \frac{B_1(u)}{B_M(u)} + \frac{B_0(u)}{B_M(u)}.$$

Thus the kernel of the difference operator $D_U^\star$ consists of the first row entries of the matrix $\Psi^{-1}$.

If $\{f_1, \dots, f_M\}$ is a basis of $U$, then

$$\Psi = \begin{pmatrix} f_1(u) & f_2(u) & \dots & f_M(u) \\ f_1(u+1) & f_2(u+1) & \dots & f_M(u+1) \\ \dots & \dots & \dots & \dots \\ f_1(u+M-1) & f_2(u+M-1) & \dots & f_M(u+M-1) \end{pmatrix}.$$

The formula for the inverse matrix elements shows that $D_U^\star$ annihilates the functions of the form $\tau_u(\mathrm{Wr}^{(\mathrm{d})}(f_1, \dots, f_{M-1}))/\mathrm{Wr}_U^{(\mathrm{d})}(u)$. Then the operator $D_U^\ddagger$ annihilates the functions of the form (3.4), since $D_U^\ddagger = D_U^\star \cdot B_M$ where $\cdot B_M$ is the operator of multiplication by the function $B_M$. $\square$

## 4. Integral transforms

### 4.1. Mellin-type transform

Let $V$ be a nondegenerate space of quasi-polynomials with data $\mathfrak{D}_V = \{n, N_i, n_{ij}, \lambda_i, m, M_a, m_{ab}, z_a\}$ where $i = 1, \dots, n$, $j = 1, \dots, N_i$, $a = 1, \dots, m$, $b = 1, \dots, M_a$. Let $V^\dagger$ be the regularized conjugate space to $V$.

For $a = 1, \dots, m$, denote by $\gamma_a$ a small circle around $z_a$ in $\mathbb{C}$ oriented counterclockwise.

Denote by $U$ the complex vector space spanned by functions of the form

$$\hat{f}_a(u) = \int\limits_{\gamma_a} x^u f(x)\, dx, \tag{4.1}$$

where $a = 1, \dots, m$, $f \in V^\dagger$. The vector space $U$ is called *bispectral dual to* $V$.

**Theorem 4.1.** *Let $V$ be a nondegenerate space of quasi-polynomials with data*

$$\mathfrak{D}_V = \{n, N_i, n_{ij}, \lambda_i, m, M_a, m_{ab}, z_a\}$$

*where $i = 1, \dots, n$, $j = 1, \dots, N_i$, $a= 1, \dots, m$, $b = 1, \dots, M_a$. Then*

(i) *The space $U$ is a nondegenerate space of quasi-exponentials with data*

$$\mathfrak{D}_U = \{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i + N_i\}$$

*where $a = 1, \dots, m$, $b = 1, \dots, M_a$, $i = 1, \dots, n$, $j = 1, \dots, N_i$.*

(ii) *Let $D_V = \sum_{i=1}^{M} \sum_{j=1}^{N} A_{ij} x^i (x\partial_x)^j$ be the regularized fundamental differential operator of $V$ where $A_{ij}$ are suitable complex numbers. Then*

$$\sum_{i=1}^{M} \sum_{j=1}^{N} A_{ij} u^j \tau_u^i$$

*is the regularized fundamental difference operator of $U$.*

The theorem is proved in Section 4.3.

### 4.2. Fourier-type transform

Let $U$ be a nondegenerate space of quasi-exponentials with data $\mathfrak{D}_U = \{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$ where $a = 1, \dots, m$, $b = 1, \dots, M_a$, $i = 1, \dots, n$, $j = 1, \dots, N_i$. Let $U^\ddagger$ be the regularized conjugate space to $U$.

For $i = 1, \dots, n$, consider the arithmetic sequence

$$\mathfrak{S}_i = \{\lambda_i - N_i, \lambda_i - N_i + 1, \lambda_i - N_i + 2, \dots, \lambda_i - N_i + n_{iN_i}\}$$

consisting of $n_{iN_i} + 1$ terms.

For $i = 1, \ldots, n$, fix a non-selfintersecting closed connected curve $\gamma_i$ in $\mathbb{C}$ oriented counterclockwise and such that the points of the sequence $\mathfrak{S}_i$ are inside $\gamma_i$ and the points of other sequences $\mathfrak{S}_j$ for $j \neq i$ are outside $\gamma_i$.

Denote by $V$ the complex vector space spanned by functions of the form

$$\hat{f}_i(x) = \int\limits_{\gamma_i} x^u f(u)\, du, \tag{4.2}$$

where $i = 1, \ldots, n$, $f \in U^{\ddagger}$. The vector space $V$ is called *bispectral dual to $U$*.

**Theorem 4.2.** *Let $U$ be a nondegenerate space of quasi-exponentials with data*

$$\mathfrak{D}_U = \{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i\}$$

*where $a = 1, \ldots, m$, $b = 1, \ldots, M_a$, $i = 1, \ldots, n$, $j = 1, \ldots, N_i$. Then*

(i) *The space $V$ is a nondegenerate space of quasi-polynomials with data*

$$\mathfrak{D}_V = \{n, N_i, n_{ij}, \lambda_i - N_i, m, M_a, m_{ab}, z_a\}$$

*where $i = 1, \ldots, n$, $j = 1, \ldots, N_i$, $a = 1, \ldots, m$, $b = 1, \ldots, M_a$.*

(ii) *Let $D_U = \sum_{i=1}^{N} \sum_{j=1}^{M} A_{ij} u^i \tau_u^j$ be the regularized fundamental difference operator of $U$ where $A_{ij}$ are suitable complex numbers. Then*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} A_{ij} x^j (x\partial_x)^i$$

*is the regularized fundamental differential operator of $V$.*

The theorem is proved in Section 4.4.

Theorems 4.1 and 4.2 imply that if $V$ is a nondegenerate space of quasi-polynomials and $U$ is bispectral dual to $V$, then $V$ is bispectral dual to $U$. Similarly, if $U$ is a nondegenerate space of quasi-exponentials and $V$ is bispectral dual to $U$, then $U$ is bispectral dual to $U$.

### 4.3. *Proof of Theorem 4.1*

The exponents of $V^{\dagger}$ at $z_a$ are

$$\{-m_{a,M_a} - 1 < \cdots < -m_{a1} - 1 < 0 < 1 < \cdots < N - M_a - 1\}.$$

Integral (4.1) is nonzero only if $f$ has a pole at $x = z_a$. If $f$ has a pole at $x = z_a$ of order $-m_{ab} - 1$, then integral (4.1) has the form $z_a^u q_{ab}(u)$ where $q_{ab}$ is a polynomial in $u$ of degree $m_{ab}$. Thus $U$ is a space of quasi-exponentials of dimension $M$ generated by quasi-exponentials $z_a^u q_{ab}(u)$ where $a = 1, \ldots, m$, $b = 1, \ldots, m_{ab}$ and $q_{ab}$ is a polynomial in $u$ of degree $m_{ab}$.

It is clear that the operator $D^\dagger = \sum_{i=1}^M \sum_{j=1}^N A_{ij} u^j \tau_u^i$ annihilates $U$. Write

$$D^\dagger = B_0(u)\tau_u^M + \cdots + B_{M-1}(u)\tau_u + B_M(u)$$

where $B_a(u)$ are polynomials in $u$ with constant coefficients. Lemma 2.3 implies that $B_0(u) = \prod_{j=1}^n (u - \lambda_j)^{N_j}$ and the polynomials $B_0, \ldots, B_M$ have no common factor of positive degree. Therefore, $D^\dagger$ is the regularized fundamental operator of $U$.

The functions $x^{\lambda_i} p_{ij}(x)$, $i = 1, \ldots, n$, $j = 1, \ldots, N_i$, form a basis of $V$. For given $i, j$, order all basis functions of $V$ except the function $x^{\lambda_i} p_{ij}(x)$. Denote by $\mathrm{Wr}_{ij}$ the Wronskian of this ordered set of $N - 1$ functions. The functions

$$f_{ij}(x) = \frac{\mathrm{Wr}_{ij}(x)}{\mathrm{Wr}_V(x) x^N \prod_{a=1}^m (x - z_a)^{M_a}}$$

form a basis in $V^\dagger$. Such a function $f_{ij}$ has the form $x^{-\lambda_i} r_{ij}$ where $r_{ij}$ is a rational function in $x$. We have

$$\mathrm{ord}_{x=0} r_{ij} = -N_i, \qquad \mathrm{ord}_{x=\infty} r_{ij} = -n_{ij} - M - 1.$$

Consider the following element of $U$,

$$F_{ij}(u) = \sum_{a=1}^m \widehat{(f_{ij})}_a(u) = \int\limits_{\bigcup_{a=1}^m \gamma_a} x^u f_{ij}(x)\, dx.$$

If $u = \lambda_i + m$ where$N_i \leqslant m \leqslant n_{ij} + M - 1$, then the integrand is a rational function with zero residues at 0 and $\infty$. Hence $F_{ij}$ is zero at $u = \lambda_i + m$ for $m$ from this arithmetic sequence. This remark together with Theorem 3.3 proves that $U$ is a nondegenerate space of quasi-exponentials with data $\{m, M_a, m_{ab}, z_a, n, N_i, n_{ij}, \lambda_i + N_i\}$.

### *4.4. Proof of Theorem 4.2*

By Lemmas 3.12 and 3.13, for any $i = 1, \ldots, n$, the functions $\hat{f}_i(x)$ in (4.2) have the form $x^{\lambda_i - N_i} p_{ij}(x)$ where $j = 1, \ldots, N_i$ and $p_{ij}$ is a polynomial of degree not greater than $n_{ij}$. Moreover, there exists a function $x^{\lambda_i - N_i} p_{ij}(x)$ with $p_{ij}$ of degree exactly equal to $n_{ij}$.

The functions $z_a^u q_{ab}(u)$, $a = 1, \ldots, m$, $b = 1, \ldots, M_a$, form a basis of $U$. For given $a, b$, order all basis functions of $U$ except the function $z_a^u q_{ab}(u)$ and denote them $f_1, \ldots, f_{M-1}$. The corresponding function

$$f_{ab}(u) = \frac{\tau_u(\mathrm{Wr}^{(\mathrm{d})}(f_1, \ldots, f_{M-1}))}{B_M(u)\mathrm{Wr}_U^{(\mathrm{d})}(u)} \in U^\ddagger$$

has the form $z_a^{-u} r_{ab}(u)$ where $r_{ab}$ is a rational function in $u$ and $\mathrm{ord}_{u=\infty} r_{ab} = M_a - m_{ab} - N - 1$. Consider the following element of $V$,

$$F_{ab}(x) = \sum_{i=1}^n \widehat{(f_{ab})}_i(x) = \int\limits_{\bigcup_{i=1}^n \gamma_i} x^u f_{ij}(u)\, du.$$

If $x = z_a$, then the integrand is a rational function in $u$ which tends to zero as $u$ tends to infinity. Denote by $^{(i)}$ the $i$th derivative. Then $F_{ab}^{(i)}(z_a) = 0$ for $i = 0, 1, \dots, N - M_a + m_{ab} - 1$, and $F_{ab}^{(N-M_a+m_{ab})}(z_a) \neq 0$.

This reason proves part (i) of the theorem.

From Theorem 3.14 and formulas for the Fourier-type integral transform, it follows that the differential operator $\sum_{i=1}^{N}\sum_{j=1}^{M} A_{ij} x^j (x\partial_x)^i$ annihilates $V$. From part (iii) of Theorem 3.10, it follows that this operator is the regularized fundamental differential operator of $V$.

## 5. Rigged spaces

In this section we consider special spaces of quasi-polynomials and quasi-exponentials with additional structures. We call such spaces rigged spaces.

### 5.1. *Spaces of quasi-polynomials of* $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$*-type*

Let $N$ be a natural number, $N > 1$. Let $\boldsymbol{n} = (n_1, \dots, n_N)$ be a vector of nonnegative integers.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$. Assume that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$.

For $i = 1, \dots, N$, let $p_i \in \mathbb{C}[x]$ be a polynomial of degree $n_i$ such that $p_i(0) \neq 0$. Denote by $V$ the complex vector space spanned by functions $x^{\lambda_i} p_i(x)$, $i = 1, \dots, N$. The dimension of $V$ is $N$.

Let $\boldsymbol{z} = (z_1, \dots, z_M)$, $M > 1$, be a subset in $\mathbb{C}$ containing all singular points of $V$. Assume that for $a = 1, \dots, M$, the set of exponents of $V$ at $z_a$ has the form

$$\{0 < 1 < \cdots < N - 2 < N - 1 + m_a\}, \quad m_a \geqslant 0.$$

We have

$$\sum_{i=1}^{N} n_i = \sum_{a=1}^{M} m_a.$$

We call the pair $(V, \boldsymbol{z})$ *a space of* $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$*-type* or *a rigged space of quasi-polynomials*.

Let $\bar{D}_V$ be the monic fundamental differential operator of $V$. The operator

$$\tilde{D}_V = x^N \prod_{a=1}^{M} (x - z_a) \bar{D}_V$$

is called *the rigged fundamental differential operator* of the rigged space $(V, \boldsymbol{z})$.

Write the rigged fundamental differential operator in the form

$$\tilde{D}_V = A_0(x)(x\partial_x)^N + A_1(x)(x\partial_x)^{N-1} + A_2(x)(x\partial_x)^{N-2} + \cdots + A_N(x).$$

Then by Lemma 2.3, all of the coefficients $A_0, \dots, A_N$ are polynomials in $x$ of degree not greater than $M$. If we write

$$\tilde{D}_V = x^M B_0(x\partial_x) + x^{M-1} B_1(x\partial_x) + \cdots + x B_{M-1}(x\partial_x) + B_M(x\partial_x),$$

where $B_i$ is a polynomial in $x\partial_x$ with constant coefficients, then

$$B_0 = \prod_{i=1}^{N}(x\partial_x - \lambda_i - n_i), \qquad B_M = (-1)^M \prod_{a=1}^{M} z_a \prod_{i=1}^{N}(x\partial_x - \lambda_i).$$

Note that a space $V$ of $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type may contain a function of the form $x^{\lambda_i}$ and be a degenerate space of quasi-polynomials in the sense of Section 2.1. Note also that the subset $\boldsymbol{z}$ may contain nonsingular points of $V$ and then the rigged fundamental differential operator $\tilde{D}_V$ differs from the regularized differential operator of the space $V$.

### 5.2. *Spaces of quasi-exponentials of* $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$*-type*

Let $M$ be a natural number, $M > 1$. Let $\boldsymbol{m} = (m_1, \ldots, m_M)$ be a vector of nonnegative integers.

Let $\boldsymbol{z} = (z_1, \ldots, z_M)$ be distinct nonzero complex numbers with fixed arguments.

For $a = 1, \ldots, M$, let $q_a \in \mathbb{C}[u]$ be a polynomial of degree $m_a$. Denote by $U$ the complex vector space spanned by the functions $z_a^u q_a(u)$, $a = 1, \ldots, M$. The dimension of $U$ is $M$.

Assume that for some natural number $N > 1$, there exists a subset of distinct numbers $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ in $\mathbb{C}$ with three properties:

- $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$.
- For $i = 1, \ldots, N$, the set of discrete exponents of $U$ at $\lambda_i$ has the form

$$\{0 < 1 < \cdots < M-2 < M-1+n_i\}, \quad n_i \geqslant 0.$$

- $\sum_{i=1}^{N} n_i = \sum_{a=1}^{M} m_a$.

In this case we call the pair $(U, \boldsymbol{\lambda})$ *a space of quasi-exponentials of* $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$*-type* or *a rigged space of quasi-exponentials*.

Let $\bar{D}_U$ be the monic fundamental difference operator of $U$. The operator

$$\tilde{D}_U = \prod_{i=1}^{N}(u - \lambda_i - n_i + 1)\bar{D}_V$$

is called *the rigged fundamental difference operator* of the rigged space $(U, \boldsymbol{\lambda})$.

Write the rigged fundamental difference operator in the form

$$\tilde{D}_U = B_0(u)\tau_u^M + B_1(u)\tau_u^{M-1} + B_2(u)\tau_u^{M-2} + \cdots + B_M(u).$$

Lemma 3.9 and Theorem 3.10 may be applied to the space $U$ and we conclude that all of the coefficients $B_0, \ldots, B_M$ are polynomials in $u$ of degree not greater than $N$ and

$$B_M = (-1)^M \prod_{a=1}^{M} z_a \prod_{i=1}^{N}(u - \lambda_i + 1).$$

If we write

$$\tilde{D}_U = u^N A_0(\tau_u) + u^{N-1} A_1(\tau_u) + \cdots + u A_{N-1}(\tau_u) + A_N(\tau_u),$$

where $A_i(\tau_u)$ is a polynomial in $\tau_u$ with constant coefficients, then

$$A_0(\tau_u) = \prod_{a=1}^{M} (\tau_u - z_a).$$

Note that a space $U$ of $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type may contain a function of the form $z_a^u$ and be not a space of quasi-exponentials in the sense of Section 3.1. Note also that the subset $\boldsymbol{\lambda}$ may contain points $\lambda_i$ with $n_i = 0$ and then the rigged fundamental difference operator $\tilde{D}_U$ differs from the regularized difference operator of the space $U$.

### 5.3. Rigged Mellin-type transform

Let $(V, \boldsymbol{z})$ be a space of quasi-polynomials of an $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type. Let $V^\star$ be the space conjugate to $V$ and defined in Section 2.4.

For $a = 1, \ldots, M$, let $\gamma_a$ be a small circle around $z_a$ in $\mathbb{C}$ oriented counterclockwise. Denote by $U$ the complex vector space spanned by the functions of the form

$$\hat{f}_a(u) = \int_{\gamma_a} x^u f(x) x^{-N} \prod_{a=1}^{M} (x - z_a)^{-1}\, dx, \tag{5.1}$$

where $a = 1, \ldots, M$, $f \in V^\star$.

**Theorem 5.1.** *Let $(V, \boldsymbol{z})$ be a space of quasi-polynomials of an $(\boldsymbol{n}, (\lambda_1, \ldots, \lambda_N), \boldsymbol{m}, (z_1, \ldots, z_M))$-type. Then*

(i) *The space $U$ is a space of quasi-exponentials of the $(\boldsymbol{m}, (z_1, \ldots, z_N), \boldsymbol{n}, (\lambda_1 + 1, \ldots, \lambda_N + 1))$-type.*

(ii) *Let $\tilde{D}_V = \sum_{i=1}^{M} \sum_{j=1}^{N} A_{ij} x^i (x\partial_x)^j$ be the rigged fundamental differential operator of $V$ where $A_{ij}$ are suitable complex numbers. Then*

$$\sum_{i=1}^{M} \sum_{j=1}^{N} A_{ij} u^j \tau_u^i$$

*is the rigged fundamental difference operator of $U$.*

The proof is similar to the proof of Theorem 4.1.

The rigged space of quasi-exponentials $(U, (\lambda_1 + 1, \ldots, \lambda_N + 1))$ is called *rigged bispectral dual* to the rigged space of quasi-polynomials $(V, \boldsymbol{z})$.

*5.4. Rigged Fourier-type transform*

Let $(U, \boldsymbol{\lambda})$ be a space of quasi-exponentials of an $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type. Let $\mathrm{Wr}_U^{(\mathrm{d})}$ be the discrete Wronskian of $U$ and let $B_M(u)$ be the last coefficient of the rigged fundamental difference operator of $U$.

The complex vector space spanned by all functions of the form

$$\frac{\tau_u(\mathrm{Wr}^{(\mathrm{d})}(f_1, \dots, f_{M-1}))}{B_M(u)\mathrm{Wr}_U^{(\mathrm{d})}(u)} \tag{5.2}$$

with $f_i \in V$ has dimension $M$. This space is denoted by $U^\bullet$ and called *rigged regularized conjugate to $U$*.

For $i = 1, \dots, N$, consider the arithmetic sequence

$$\mathfrak{S}_i = \{\lambda_i - 1, \lambda_i, \lambda_i + 1, \dots, \lambda_i + n_i - 1\}$$

consisting of $n_i + 1$ terms.

For $i = 1, \dots, N$, fix a non-selfintersecting closed connected curve $\gamma_i$ in $\mathbb{C}$ oriented counterclockwise such that it encircles the sequence $\mathfrak{S}_i$ and does not contain inside or intersect with points of other sequences $\mathfrak{S}_j$ for $j \neq i$.

Denote by $V$ the complex vector space spanned by functions of the form

$$\hat{f}_i(x) = \int\limits_{\gamma_i} x^u f(u)\, du,$$

where $i = 1, \dots, n$, $f \in U^\bullet$.

**Theorem 5.2.** *Let $(U, \boldsymbol{\lambda})$ be a space of quasi-exponentials of an $(\boldsymbol{m}, (z_1, \dots, z_M), \boldsymbol{n}, (\lambda_1, \dots, \lambda_N))$-type. Then*

(i) *The space $V$ is a space of quasi-exponentials of the $(\boldsymbol{n}, (\lambda_1 - 1, \dots, \lambda_N - 1), \boldsymbol{m}, (z_1, \dots, z_N))$-type.*
(ii) *Let $\tilde{D}_u = \sum_{i=1}^M \sum_{j=1}^N A_{ij} u^i \tau_u^j$ be the rigged fundamental difference operator of $U$ where $A_{ij}$ are suitable complex numbers. Then*

$$\sum_{i=1}^M \sum_{j=1}^N A_{ij} x^j (x\partial_x)^i$$

*is the rigged fundamental differential operator of $V$.*

The proof is similar to the proof of Theorem 4.2.

The rigged space of quasi-polynomials $(V, \boldsymbol{z})$ is called *rigged bispectral dual* to the rigged space of quasi-exponentials $(U, \boldsymbol{\lambda})$.

Theorems 5.1 and 5.2 imply that if $V$ is a rigged space of quasi-polynomials and $U$ is rigged bispectral dual to $V$, then $V$ is rigged bispectral dual to $U$. Similarly, if $U$ is a rigged space of quasi-exponentials and $V$ is rigged bispectral dual to $U$, then $U$ is rigged bispectral dual to $U$.

## 6. Rigged spaces and solutions of the Bethe ansatz equations

### 6.1. Critical points of master functions and rigged spaces of quasi-polynomials

Let $(V, z)$ be a space of quasi-polynomials of an $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type. We construct the associated master function as follows. Set

$$\bar{n}_i = n_{i+1} + \cdots + n_N, \quad i = 1, \ldots, N-1.$$

Consider the new $\bar{n}_1 + \cdots + \bar{n}_{N-1}$ auxiliary variables

$$\boldsymbol{t}^{\langle \boldsymbol{n} \rangle} = \big(t_1^{(1)}, \ldots, t_{\bar{n}_1}^{(1)}, t_1^{(2)}, \ldots, t_{\bar{n}_2}^{(2)}, \ldots, t_1^{(N-1)}, \ldots, t_{\bar{n}_{N-1}}^{(N-1)}\big).$$

Define *the master function*

$$\begin{aligned}\Phi\big(\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}; \boldsymbol{\lambda}; \boldsymbol{m}; \boldsymbol{z}\big) = {} & \prod_{a=1}^{M} z_a^{m_a(\lambda_1 + m_a/2)} \prod_{1 \leqslant a < b \leqslant M} (z_a - z_b)^{m_a m_b} \prod_{i=1}^{N} \prod_{j=1}^{\bar{n}_i} \big(t_j^{(i)}\big)^{\lambda_{i+1} - \lambda_i + 1} \\ & \times \prod_{a=1}^{M} \prod_{j=1}^{\bar{n}_1} \big(t_j^{(1)} - z_a\big)^{-m_a} \prod_{i=1}^{N-1} \prod_{j<j'} \big(t_j^{(i)} - t_{j'}^{(i)}\big)^2 \\ & \times \prod_{i=1}^{N-2} \prod_{j=1}^{\bar{n}_i} \prod_{j'=1}^{\bar{n}_{i+1}} \big(t_j^{(i)} - t_{j'}^{(i+1)}\big)^{-1}. \end{aligned} \tag{6.1}$$

The master function is symmetric with respect to the group $\boldsymbol{\Sigma}_{\bar{\boldsymbol{n}}} = \Sigma_{\bar{n}_1} \times \cdots \times \Sigma_{\bar{n}_{N-1}}$ of permutations of variables $t_j^{(i)}$ preserving the upper index.

A point $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}$ with complex coordinates is called *a critical point* of $\Phi(\cdot; \boldsymbol{\lambda}; \boldsymbol{m}; \boldsymbol{z})$ if

$$\left(\Phi^{-1} \frac{\partial \Phi}{\partial t_j^{(i)}}\right)\big(\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}; \boldsymbol{\lambda}; \boldsymbol{m}; \boldsymbol{z}\big) = 0, \quad i = 1, \ldots, N-1, \ j = 1, \ldots, \bar{n}_i.$$

In other words, a point $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}$ is a critical point if the following system of $\bar{n}_1 + \cdots + \bar{n}_{N-1}$ equations is satisfied

$$\begin{aligned} \frac{\lambda_1 - \lambda_2 - 1}{t_j^{(1)}} + \sum_{a=1}^{M} \frac{m_a}{t_j^{(1)} - z_a} - \sum_{j'=1,\, j' \neq j}^{\bar{n}_1} \frac{2}{t_j^{(1)} - t_{j'}^{(1)}} + \sum_{j'=1}^{\bar{n}_2} \frac{1}{t_j^{(1)} - t_{j'}^{(2)}} &= 0, \\ \frac{\lambda_i - \lambda_{i+1} - 1}{t_j^{(i)}} - \sum_{j'=1,\, j' \neq j}^{\bar{n}_i} \frac{2}{t_j^{(i)} - t_{j'}^{(i)}} + \sum_{j'=1}^{\bar{n}_{i-1}} \frac{1}{t_j^{(i)} - t_{j'}^{(i-1)}} + \sum_{j'=1}^{\bar{n}_{i+1}} \frac{1}{t_j^{(i)} - t_{j'}^{(i+1)}} &= 0, \\ \frac{\lambda_{N-1} - \lambda_N - 1}{t_j^{(N-1)}} - \sum_{j'=1,\, j' \neq j}^{\bar{n}_{N-1}} \frac{2}{t_j^{(N-1)} - t_{j'}^{(N-1)}} + \sum_{j'=1}^{\bar{n}_{N-2}} \frac{1}{t_j^{(N-1)} - t_{j'}^{(N-2)}} &= 0, \end{aligned} \tag{6.2}$$

where $j = 1, \ldots, \bar{n}_1$ in the first group of equations, $i = 2, \ldots, N-2$ and $j = 1, \ldots, \bar{n}_i$ in the second group of equations, $j = 1, \ldots, \bar{n}_{N-1}$ in the last group of equations.

In the Gaudin model, Eqs. (6.2) are called *the Gaudin Bethe ansatz equations*.

The $\boldsymbol{\Sigma}_{\bar{\boldsymbol{n}}}$-orbit of a point $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle} \in \mathbb{C}^{\bar{n}_1 + \cdots + \bar{n}_{N-1}}$ is uniquely determined by the $N-1$-tuple $\boldsymbol{y}^{\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}} = (y_1, \ldots, y_{N-1})$ of polynomials in $x$, where

$$y_i = \prod_{j=1}^{\bar{n}_i} \bigl(x - t_j^{(i)}\bigr), \quad i = 1, \ldots, N-1.$$

We say that $\boldsymbol{y}$ *represents* the orbit. Each polynomial of the tuple is considered up to multiplication by a nonzero number since we are interested in the roots of the polynomial only.

We say that $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle} \in \mathbb{C}^{\bar{n}_1 + \cdots + \bar{n}_{N-1}}$ is *Gaudin admissible* if the value $\Phi(\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}; \boldsymbol{\lambda}; \boldsymbol{m}; \boldsymbol{z})$ is well defined and is not zero.

A point $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}$ is Gaudin admissible if and only if the associated tuple has the following properties.

- For $a = 1, \ldots, M$, if $m_a > 0$, then $y_1(z_a) \neq 0$.
- For all $i$, $y_i(0) \neq 0$.
- For all $i$, the polynomial $y_i$ has no multiple roots and no common roots with $y_{i-1}$ or $y_{i+1}$.

Such tuples are called *Gaudin admissible*.

Return to $(V, \boldsymbol{z})$, a space of an $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type.

The space $V = \langle x^{\lambda_1} p_1(x), \ldots, x^{\lambda_N} p_N(x) \rangle$ determines the $N-1$-tuple $\boldsymbol{y}^V = (y_1, \ldots, y_{N-1})$ of polynomials in $x$, where

$$y_{N-1} = p_N, \qquad y_i = x^{\frac{(N-i)(N-i-1)}{2} - \lambda_{i+1} - \cdots - \lambda_N} \operatorname{Wr}\bigl(x^{\lambda_{i+1}} p_{i+1}(x), \ldots, x^{\lambda_N} p_N(x)\bigr)$$

for $i = 1, \ldots, N-2$.

We call the rigged space $(V, \boldsymbol{z})$ *Gaudin admissible* if the tuple $\boldsymbol{y}^V$ is Gaudin admissible.

**Theorem 6.1.** *(See [15,17].)*

(i) *Assume that the rigged space $(V, \boldsymbol{z})$ is Gaudin admissible. Then the tuple $\boldsymbol{y}^V$ represents the orbit of a critical point of the master function.*

(ii) *Assume that $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}$ is Gaudin admissible and $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}$ is a critical point of the master function. Let $\boldsymbol{y} = (y_1, \ldots, y_{N-1})$ be the tuple representing the orbit of $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}$. Then the differential operator*

$$\bar{D} = \left(\partial_x - \ln'\left(\frac{x^{\lambda_1 - N + 1} \prod_{a=1}^M (x - z_a)^{m_a}}{y_1}\right)\right)\left(\partial_x - \ln'\left(\frac{x^{\lambda_2 - N + 2} y_1}{y_2}\right)\right)$$
$$\ldots \left(\partial_x - \ln'\left(\frac{x^{\lambda_{N-1} - 1} y_{N-2}}{y_{N-1}}\right)\right)\bigl(\partial_x - \ln'\bigl(x^{\lambda_N} y_{N-1}\bigr)\bigr)$$

*of order $N$ is the monic fundamental differential operator of a Gaudin admissible rigged space of quasi-polynomials $(V, \boldsymbol{z})$ of the $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type.*

(iii) *The correspondence between Gaudin admissible rigged spaces of quasi-polynomials of the* $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$*-type and orbits of Gaudin admissible critical points of the master function described in parts* (i), (ii) *is reflexive.*

This theorem establishes a one-to-one correspondence between Gaudin admissible rigged spaces of quasi-polynomials of $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type and orbits of Gaudin admissible critical points of the master function.

### 6.2. Solutions of the Bethe ansatz equations and rigged spaces of quasi-exponentials

Let $(U, \boldsymbol{\lambda})$ be a space of quasi-exponentials of an $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type. We define the associated system of Bethe ansatz equations as follows. Set

$$\bar m_a = m_{a+1} + \cdots + m_M, \quad a = 1, \ldots, M-1.$$

Consider the new $\bar m_1 + \cdots + \bar m_{M-1}$ auxiliary variables

$$\boldsymbol{t}^{\langle \boldsymbol{m} \rangle} = \big(t_1^{(1)}, \ldots, t_{\bar m_1}^{(1)}, t_1^{(2)}, \ldots, t_{\bar m_2}^{(2)}, \ldots, t_1^{(M-1)}, \ldots, t_{\bar m_{M-1}}^{(M-1)}\big).$$

*The Bethe ansatz equations of the XXX model* is the following system of $\bar m_1 + \cdots + \bar m_{M-1}$ equations:

$$\begin{gathered}
\prod_{i=1}^{N} \frac{t_b^{(1)} - \lambda_i}{t_b^{(1)} - \lambda_i - n_i} \prod_{b' \neq b} \frac{t_b^{(1)} - t_{b'}^{(1)} - 1}{t_b^{(1)} - t_{b'}^{(1)} + 1} \prod_{b'=1}^{\bar m_2} \frac{t_b^{(1)} - t_{b'}^{(2)} + 1}{t_b^{(1)} - t_{b'}^{(2)}} = \frac{z_2}{z_1}, \\
\prod_{b'=1}^{\bar m_{a-1}} \frac{t_b^{(a)} - t_{b'}^{(a-1)}}{t_b^{(a)} - t_{b'}^{(a-1)} - 1} \prod_{b' \neq b} \frac{t_b^{(a)} - t_{b'}^{(a)} - 1}{t_b^{(a)} - t_{b'}^{(a)} + 1} \prod_{b'=1}^{\bar m_{a+1}} \frac{t_b^{(a)} - t_{b'}^{(a+1)} + 1}{t_b^{(a)} - t_{b'}^{(a+1)}} = \frac{z_{a+1}}{z_a}, \\
\prod_{b'=1}^{\bar m_{N-2}} \frac{t_b^{(N-1)} - t_{b'}^{(N-2)}}{t_b^{(N-1)} - t_{b'}^{(N-2)} - 1} \prod_{b' \neq b} \frac{t_b^{(N-1)} - t_{b'}^{(N-1)} - 1}{t_b^{(N-1)} - t_{b'}^{(N-1)} + 1} = \frac{z_M}{z_{M-1}}
\end{gathered} \tag{6.3}$$

where $b = 1, \ldots, \bar m_1$ in the first group of equations, $a = 2, \ldots, M-2$ and $b = 1, \ldots, \bar m_a$ in the second group of equations, $b = 1, \ldots, \bar m_{M-1}$ in the last group of equations.

The system of Bethe ansatz equations is symmetric with respect to the group $\boldsymbol{\Sigma}_{\bar{\boldsymbol{m}}} = \Sigma_{\bar m_1} \times \cdots \times \Sigma_{\bar m_{M-1}}$ of permutations of variables $t_b^{(a)}$ preserving the upper index.

The $\boldsymbol{\Sigma}_{\bar{\boldsymbol{m}}}$-orbit of a point $\boldsymbol{t}^{\langle \boldsymbol{m} \rangle} \in \mathbb{C}^{\bar m_1 + \cdots + \bar m_{M-1}}$ is uniquely determined by the $M-1$-tuple $\boldsymbol{y}^{\boldsymbol{t}^{\langle \boldsymbol{m} \rangle}} = (y_1, \ldots, y_{M-1})$ of polynomials in $u$, where

$$y_a = \prod_{b=1}^{\bar m_a} \big(u - t_b^{(a)}\big), \quad a = 1, \ldots, M-1.$$

We say that $\boldsymbol{y}$ *represents* the orbit. Each polynomial of the tuple is considered up to multiplication by a nonzero number.

We say that $\boldsymbol{t}^{\langle \boldsymbol{m} \rangle} \in \mathbb{C}^{\bar m_1 + \cdots + \bar m_{M-1}}$ is *XXX admissible* of $\boldsymbol{\lambda}$-type if

$$t_b^{(a)} \neq t_{b'}^{(a)}, \qquad t_b^{(a)} \neq t_{b'}^{(a)} + 1, \qquad t_b^{(a)} \neq t_{b'}^{(a+1)}, \qquad t_b^{(a)} \neq t_{b'}^{(a+1)} - 1,$$
$$t_b^{(1)} \neq \lambda_i + r,$$

for all $a, b, b', i$, and $r = 0, \dots, n_i$.

If $\boldsymbol{t}^{\langle \boldsymbol{m} \rangle} \in \mathbb{C}^{\bar{m}_1 + \cdots + \bar{m}_{M-1}}$ is XXX admissible, then the corresponding tuple $\boldsymbol{y}^{\boldsymbol{t}^{\langle \boldsymbol{m} \rangle}}$ is called *XXX admissible*.

Return to $(U, \boldsymbol{\lambda})$, a rigged space of quasi-exponentials of an $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type.

The space $U = \langle z_1^u q_1(u), \dots, z_M^u q_M(u) \rangle$ determines the $M-1$-tuple $\boldsymbol{y}^U = (y_1, \dots, y_{M-1})$ of polynomials in $u$, where

$$y_{M-1} = q_M, \qquad y_a = \prod_{b=a+1}^{M} z_b^{-u} \mathrm{Wr}^{(\mathrm{d})}\big(z_{a+1}^u q_{a+1}(u), \dots, z_M^u q_M(u)\big)$$

for $a = 1, \dots, M-2$.

We call the rigged space $(U, \boldsymbol{\lambda})$ *XXX admissible* if the tuple $\boldsymbol{y}^U$ is XXX admissible.

**Theorem 6.2.** *(See [14,17].)*

(i) *Assume that the rigged space $(U, \boldsymbol{\lambda})$ is XXX admissible. Then the tuple $\boldsymbol{y}^U$ represents the orbit of a solution of the XXX Bethe ansatz equations* (6.3).

(ii) *Assume that $\boldsymbol{t}^{\langle \boldsymbol{m} \rangle}$ is XXX admissible and $\boldsymbol{t}^{\langle \boldsymbol{m} \rangle}$ is a solution of the XXX Bethe ansatz equations* (6.3). *Let $\boldsymbol{y} = (y_1, \dots, y_{M-1})$ be the tuple representing the orbit of $\boldsymbol{t}^{\langle \boldsymbol{m} \rangle}$. Then the difference operator*

$$\bar{D} = \left(\tau_u - \frac{y_1(u)}{y_1(u+1)} \prod_{i=1}^{N} \frac{u - \lambda_i + 1}{u - \lambda_i - n_i + 1} z_1\right)\left(\tau_u - \frac{y_1(u+1)}{y_1(u)} \frac{y_2(u)}{y_2(u+1)} z_2\right)$$
$$\dots \left(\tau_u - \frac{y_{M-2}(u+1)}{y_{M-2}(u)} \frac{y_{M-1}(u)}{y_{M-1}(u+1)} z_{M-1}\right)\left(\tau_u - \frac{y_{M-1}(u+1)}{y_{M-1}(u)} z_M\right)$$

*of order $M$ is the monic fundamental difference operator of a rigged space of quasi-exponentials $(U, \boldsymbol{\lambda})$ of $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type.*

(iii) *The correspondence between XXX admissible rigged spaces of quasi-exponentials of the $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type and orbits of XXX admissible solutions of the XXX Bethe ansatz equations described in parts* (i), (ii) *is reflexive.*

This theorem establishes a one-to-one correspondence between XXX admissible rigged spaces of quasi-exponentials of $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type and orbits of XXX admissible solutions of the XXX Bethe ansatz equations.

## 7. Finiteness of solutions of the Bethe ansatz equations

### *7.1. Finiteness of the Gaudin admissible critical points*

**Lemma 7.1.** *For given fixed $\boldsymbol{m}$, $\boldsymbol{z}$ and generic $\boldsymbol{\lambda}$, the master function $\Phi(\boldsymbol{t}^{\langle \boldsymbol{n} \rangle}; \boldsymbol{\lambda}; \boldsymbol{m}; \boldsymbol{z})$, defined in* (6.1)*, has only finitely many Gaudin admissible critical points.*

The lemma follows from Lemma 2.1 in [15].

*7.2. The number of orbits of the Gaudin admissible critical points*

Let $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ be distinct numbers such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$. Set $\bar{n} = n_1 + \dots + n_N$. Consider the complex vector space $X$ spanned by functions $x^{\lambda_i + j}$, $i = 1, \dots, N$, $j = 0, \dots, n_i$. The space $X$ is of dimension $\bar{n} + N$.

For $z \in \mathbb{C}^*$, define a complete flag $\boldsymbol{F}(z)$ in $X$,

$$\boldsymbol{F}(z) = \{0 = F_0(z) \subset F_1(z) \subset \dots \subset F_{\bar{n}+N}(z) = X\},$$

where $F_k(z)$ consists of all $f \in X$ which have zero at $z$ of order not less that $\bar{n} + N - k$. The subspace $F_k(z)$ has dimension $k$.

Define two complete flags of $X$ *at infinity*.

Say that $x^{\lambda_i + j} <_1 x^{\lambda_{i'} + j'}$ if $i < i'$ or $i = i'$ and $j < j'$. Set

$$\boldsymbol{F}(\infty_1) = \{0 = F_0(\infty_1) \subset F_1(\infty_1) \subset \dots \subset F_{\bar{n}+N}(\infty_1) = X\},$$

where $F_k(\infty_1)$ is spanned by $k$ smallest elements with respect to $<_1$.

Say that $x^{\lambda_i + j} <_2 x^{\lambda_{i'} + j'}$ if $i > i'$ or $i = i'$ and $j < j'$. Set

$$\boldsymbol{F}(\infty_2) = \{0 = F_0(\infty_2) \subset F_1(\infty_2) \subset \dots \subset F_{\bar{n}+N}(\infty_2) = X\},$$

where $F_k(\infty_2)$ is spanned by $k$ smallest elements with respect to $<_2$.

Denote by $\mathrm{Gr}(X, N)$ the Grassmannian manifold of $N$-dimensional vector subspaces of $X$. Let $\boldsymbol{F}$ be a complete flag of $X$,

$$\boldsymbol{F} = \{0 = F_0 \subset F_1 \subset \dots \subset F_{\bar{n}+N} = X\}.$$

*A ramification sequence* is a sequence $(c_1, \dots, c_N) \in \mathbb{Z}^N$ such that $\bar{n} \geqslant c_1 \geqslant \dots \geqslant c_N \geqslant 0$. For a ramification sequence $\boldsymbol{c} = (c_1, \dots, c_N)$ define *the Schubert cell*

$$\begin{aligned} \Omega^o_{\boldsymbol{c}}(\boldsymbol{F}) = \{ V \in \mathrm{Gr}(X, N) \mid \dim(V \cap F_u) = \ell,& \\ \bar{n} + \ell - c_\ell \leqslant u < \bar{n} + \ell + 1 - c_{\ell+1},\ \ell = 0, \dots, N &\}, \end{aligned}$$

where $c_0 = \bar{n}$, $c_{N+1} = 0$. The cell $\Omega^o_{\boldsymbol{c}}(\boldsymbol{F})$ is a smooth connected variety. The closure of $\Omega^o_{\boldsymbol{c}}(\boldsymbol{F})$ is denoted by $\Omega_{\boldsymbol{c}}(\boldsymbol{F})$. The codimension of $\Omega^o_{\boldsymbol{c}}(\boldsymbol{F})$ is

$$|\boldsymbol{c}| = c_1 + c_2 + \dots + c_N.$$

Every $N$-dimensional vector subspace of $X$ belongs to a unique Schubert cell $\Omega^o_{\boldsymbol{c}}(\boldsymbol{F})$.

For $a = 1, \dots, M$, define the ramification sequence

$$\boldsymbol{c}(a) = (m_a, 0, \dots, 0).$$

Define the ramification sequences

$$\boldsymbol{c}(\infty_1) = (n_2 + \cdots + n_N, n_3 + \cdots + n_N, \ldots, n_N, 0),$$
$$\boldsymbol{c}(\infty_2) = (n_1 + \cdots + n_{N-1}, n_1 + \cdots + n_{N-2}, \ldots, n_1, 0).$$

**Lemma 7.2.**

- *We have*

$$\sum_{a=1}^{M} \operatorname{codim} \Omega^o_{\boldsymbol{c}(a)}\big(\boldsymbol{F}(z_a)\big) + \operatorname{codim} \Omega^o_{\boldsymbol{c}(\infty_1)}\big(\boldsymbol{F}(\infty_1)\big) + \operatorname{codim} \Omega^o_{\boldsymbol{c}(\infty_2)}\big(\boldsymbol{F}(\infty_2)\big)$$
$$= \dim \operatorname{Gr}(X, N) = N\bar{n}.$$

- *Let $V \in \operatorname{Gr}(X, N)$. The pair $(V, z)$ is a space of the $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type, if and only if $V$ belongs to the intersection of $M + 2$ Schubert cells*

$$\Omega^o_{\boldsymbol{c}(1)}\big(\boldsymbol{F}(z_1)\big) \cap \Omega^o_{\boldsymbol{c}(2)}\big(\boldsymbol{F}(z_2)\big) \cap \cdots$$
$$\cap\, \Omega^o_{\boldsymbol{c}(M)}\big(\boldsymbol{F}(z_M)\big) \cap \Omega^o_{\boldsymbol{c}(\infty_1)}\big(\boldsymbol{F}(\infty_1)\big) \cap \Omega^o_{\boldsymbol{c}(\infty_2)}\big(\boldsymbol{F}(\infty_2)\big).$$

The proof is straightforward.

According to Schubert calculus, the multiplicity of the intersection of Schubert cycles

$$\Omega_{\boldsymbol{c}(1)}\big(\boldsymbol{F}(z_1)\big) \cap \Omega_{\boldsymbol{c}(2)}\big(\boldsymbol{F}(z_2)\big) \cap \cdots$$
$$\cap\, \Omega_{\boldsymbol{c}(M)}\big(\boldsymbol{F}(z_M)\big) \cap \Omega_{\boldsymbol{c}(\infty_1)}\big(\boldsymbol{F}(\infty_1)\big) \cap \Omega_{\boldsymbol{c}(\infty_2)}\big(\boldsymbol{F}(\infty_2)\big) \tag{7.1}$$

can be expressed in representation-theoretic terms as follows.

For a ramification sequence $\boldsymbol{c}$ denote by $L^{\langle N\rangle}_{\boldsymbol{c}}$ the finite-dimensional irreducible $\mathfrak{gl}_N$-module with highest weight $\boldsymbol{c}$. Any $\mathfrak{gl}_N$-module $L^{\langle N\rangle}$ has a natural structure of an $\mathfrak{sl}_N$-module denoted by $\widetilde{L}^{\langle N\rangle}$. By [5], the multiplicity of the intersection in (7.1) is equal to the multiplicity of the trivial $\mathfrak{sl}_N$-module in the tensor product of $\mathfrak{sl}_N$-modules

$$\widetilde{L}^{\langle N\rangle}_{\boldsymbol{c}(1)} \otimes \cdots \otimes \widetilde{L}^{\langle N\rangle}_{\boldsymbol{c}(M)} \otimes \widetilde{L}^{\langle N\rangle}_{\boldsymbol{c}(\infty_1)} \otimes \widetilde{L}^{\langle N\rangle}_{\boldsymbol{c}(\infty_2)}. \tag{7.2}$$

**Proposition 7.3.** *(See [11].) The multiplicity of the trivial $\mathfrak{sl}_N$-module in the tensor product* (7.2) *is equal to the dimension of the weight subspace of weight* $[n_1, \ldots, n_N]$ *in the tensor product of $\mathfrak{gl}_N$-modules*

$$L^{\langle N\rangle}_{\boldsymbol{c}(1)} \otimes \cdots \otimes L^{\langle N\rangle}_{\boldsymbol{c}(M)}. \tag{7.3}$$

**Corollary 7.4.** *For generic $\boldsymbol{\lambda}$, the number of orbits of the Gaudin admissible critical points of the master function $\Phi(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}; \boldsymbol{\lambda}; \boldsymbol{m}; \boldsymbol{z})$, is not greater than the dimension of the weight space* $(L^{\langle N\rangle}_{\boldsymbol{c}(1)} \otimes \cdots \otimes L^{\langle N\rangle}_{\boldsymbol{c}(M)})[n_1, \ldots, n_N]$.

*7.3. Finiteness of solutions of Bethe ansatz equations (6.3)*

Let $\boldsymbol{z} = (z_1, \dots, z_M) \in \mathbb{C}^M$. Let $(\bar{m}_1, \dots, \bar{m}_{M-1})$ be a collection of nonnegative integers. We say that $\boldsymbol{z}$ *is separating with respect to* $(\bar{m}_1, \dots, \bar{m}_{M-1})$ if

$$\prod_{a=1}^{M-1} \left(\frac{z_{a+1}}{z_a}\right)^{c_a} \neq 1$$

for all sets of integers $\{c_1, \dots, c_{M-1}\}$ such that $0 \leqslant c_a \leqslant \bar{m}_a$, $\sum_a c_a > 0$.

Clearly, for given $(\bar{m}_1, \dots, \bar{m}_{M-1})$, a generic $\boldsymbol{z}$ is separating.

**Lemma 7.5.** *Let $\boldsymbol{n}$, $\boldsymbol{\lambda}$, $(\bar{m}_1, \dots, \bar{m}_{M-1})$ be fixed. Let $\boldsymbol{z}$ be separating with respect to $(\bar{m}_1, \dots, \bar{m}_{M-1})$. Then the Bethe ansatz equations* (6.3) *have only finitely many XXX admissible solutions of $\boldsymbol{\lambda}$-type.*

**Proof.** If the algebraic set of XXX admissible solutions of (6.3) is infinite, then it is unbounded. Suppose that we have a sequence of solutions which is unbounded. Without loss of generality, we assume that $t_b^{(a)}$ tends to infinity for $a = 1, \dots, M-1$, $b = 1, \dots, c_a$, and remains bounded for all other values of $a, b$. Multiply all of the equations in (6.3) corresponding to a variables $t_b^{(a)}$ with $a = 1, \dots, M-1$, $b = 1, \dots, c_a$, and pass in the product to the limit along our sequence of solutions. Then the resulting equation is

$$\prod_{a=1}^{M-1} \left(\frac{z_{a+1}}{z_a}\right)^{c_a} = 1.$$

This equation contradicts to our assumption. $\square$

*7.4. The number of orbits of the XXX admissible solutions*

Let $z_1, \dots, z_M \in \mathbb{C}$ be distinct numbers with fixed argument. Set $\bar{m} = m_1 + \dots + m_M$. Consider the complex vector space $Y$ spanned by functions $z_a^u u^b$, $a = 1, \dots, M$, $b = 0, \dots, m_a$. The space $Y$ is of dimension $\bar{m} + M$.

For $\lambda \in \mathbb{C}$, define a complete flag $\boldsymbol{F}(\lambda)$ in $Y$,

$$\boldsymbol{F}(\lambda) = \left\{0 = F_0(\lambda) \subset F_1(\lambda) \subset \dots \subset F_{\bar{m}+M}(\lambda) = Y\right\},$$

where $F_k(\lambda)$ consists of all $f \in Y$ which are divisible by $\prod_{j=1}^{\bar{m}+M-k}(u - \lambda - j + 1)$. The subspace $F_k(\lambda)$ has dimension $k$.

Define two complete flags of $Y$ *at infinity*.

Say that $z_a^u u^b <_1 z_{a'}^u u^{b'}$ if $a < a'$ or $a = a'$ and $b < b'$. Set

$$\boldsymbol{F}(\infty_1) = \left\{0 = F_0(\infty_1) \subset F_1(\infty_1) \subset \dots \subset F_{\bar{m}+M}(\infty_1) = Y\right\},$$

where $F_k(\infty_1)$ is spanned by $k$ smallest elements with respect to $<_1$.

Say that $z_a^u u^b <_2 z_{a'}^u u^{b'}$ if $a > a'$ or $a = a'$ and $b < b'$. Set

$$\boldsymbol{F}(\infty_2) = \{0 = F_0(\infty_2) \subset F_1(\infty_2) \subset \cdots \subset F_{\bar{m}+M}(\infty_2) = Y\},$$

where $F_k(\infty_2)$ is spanned by $k$ smallest elements with respect to $<_2$.

Denote by $\mathrm{Gr}(Y, M)$ the Grassmannian manifold of $M$-dimensional vector subspaces of $Y$. Let $\boldsymbol{F}$ be a complete flag of $Y$,

$$\boldsymbol{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_{\bar{m}+M} = Y\}.$$

For a ramification sequence $\boldsymbol{c} = (c_1, \ldots, c_M) \in \mathbb{Z}^M$, $\bar{m} \geqslant c_1 \geqslant \cdots \geqslant c_M \geqslant 0$, denote by $\Omega_{\boldsymbol{c}}^o(\boldsymbol{F}) \subset \mathrm{Gr}(Y, M)$ the corresponding Schubert cell.

For $i = 1, \ldots, N$, define the ramification sequence

$$\boldsymbol{c}(i) = (n_i, 0, \ldots, 0).$$

Define the ramification sequences

$$\begin{aligned}
\boldsymbol{c}(\infty_1) &= (m_2 + \cdots + m_M, m_3 + \cdots + m_M, \ldots, m_M, 0),\\
\boldsymbol{c}(\infty_2) &= (m_1 + \cdots + m_{M-1}, m_1 + \cdots + m_{M-2}, \ldots, m_1, 0).
\end{aligned}$$

**Lemma 7.6.**

- *We have*

$$\sum_{i=1}^{N} \operatorname{codim} \Omega_{\boldsymbol{c}(i)}^o\big(\boldsymbol{F}(\lambda_i)\big) + \operatorname{codim} \Omega_{\boldsymbol{c}(\infty_1)}^o\big(\boldsymbol{F}(\infty_1)\big) + \operatorname{codim} \Omega_{\boldsymbol{c}(\infty_2)}^o\big(\boldsymbol{F}(\infty_2)\big) = \dim \mathrm{Gr}(Y, M) = M\bar{m}.$$

- *Let $U \in \mathrm{Gr}(Y, M)$. The pair $(U, \boldsymbol{\lambda})$ is a space of the $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type, if and only if $U$ belongs to the intersection of $N + 2$ Schubert cells*

$$\Omega_{\boldsymbol{c}(1)}^o\big(\boldsymbol{F}(\lambda_1)\big) \cap \Omega_{\boldsymbol{c}(2)}^o\big(\boldsymbol{F}(\lambda_2)\big) \cap \cdots \cap \Omega_{\boldsymbol{c}(M)}^o\big(\boldsymbol{F}(\lambda_N)\big) \cap \Omega_{\boldsymbol{c}(\infty_1)}^o\big(\boldsymbol{F}(\infty_1)\big) \cap \Omega_{\boldsymbol{c}(\infty_2)}^o\big(\boldsymbol{F}(\infty_2)\big).$$

The proof is straightforward.

According to Schubert calculus, the multiplicity of the intersection of Schubert cycles

$$\Omega_{\boldsymbol{c}(1)}\big(\boldsymbol{F}(\lambda_1)\big) \cap \Omega_{\boldsymbol{c}(2)}\big(\boldsymbol{F}(\lambda_2)\big) \cap \cdots \cap \Omega_{\boldsymbol{c}(M)}\big(\boldsymbol{F}(\lambda_N)\big) \cap \Omega_{\boldsymbol{c}(\infty_1)}\big(\boldsymbol{F}(\infty_1)\big) \cap \Omega_{\boldsymbol{c}(\infty_2)}\big(\boldsymbol{F}(\infty_2)\big) \tag{7.4}$$

can be expressed in representation-theoretic terms as follows.

For a ramification sequence $\boldsymbol{c} = (c_1, \ldots, c_M) \in \mathbb{Z}^M$, denote by $L_{\boldsymbol{c}}^{\langle M\rangle}$ the finite-dimensional irreducible $\mathfrak{gl}_M$-module with highest weight $\boldsymbol{c}$. Any $\mathfrak{gl}_M$-module $L^{\langle M\rangle}$ has a natural structure of an $\mathfrak{sl}_M$-module denoted by $\widetilde{L}^{\langle M\rangle}$. By [5], the multiplicity of the intersection in (7.1) is equal to the multiplicity of the trivial $\mathfrak{sl}_M$-module in the tensor product of $\mathfrak{sl}_M$-modules

$$\widetilde{L}^{\langle M\rangle}_{\boldsymbol{c}(1)} \otimes \cdots \otimes \widetilde{L}^{\langle M\rangle}_{\boldsymbol{c}(N)} \otimes \widetilde{L}^{\langle M\rangle}_{\boldsymbol{c}(\infty_1)} \otimes \widetilde{L}^{\langle M\rangle}_{\boldsymbol{c}(\infty_2)}. \tag{7.5}$$

By Proposition 7.3, the multiplicity of the trivial $\mathfrak{sl}_M$-module in the tensor product (7.5) is equal to the dimension of the weight subspace of weight $[m_1, \dots, m_M]$ in the tensor product of $\mathfrak{gl}_M$-modules

$$L^{\langle M\rangle}_{\boldsymbol{c}(1)} \otimes \cdots \otimes L^{\langle M\rangle}_{\boldsymbol{c}(N)}. \tag{7.6}$$

**Corollary 7.7.** *For generic $z$, the number of orbits of the XXX admissible of $\boldsymbol{\lambda}$-type solutions of the Bethe ansatz equations* (6.3) *is not greater than the dimension of the weight space* $(L^{\langle M\rangle}_{\boldsymbol{c}(1)} \otimes \cdots \otimes L^{\langle M\rangle}_{\boldsymbol{c}(N)})[m_1, \dots, m_M]$.

The proof is straightforward.

## 8. The KZ and dynamical Hamiltonians

### 8.1. The Gaudin KZ Hamiltonians

Let $E_{ij}$, $i, j = 1, \dots, N$, be the standard generators of the complex Lie algebra $\mathfrak{gl}_N$.
Let

$$\Omega^0 = \frac{1}{2}\sum_{i=1}^{N} E_{ii} \otimes E_{ii}, \qquad \Omega^+ = \Omega^0 + \sum_{i<j} E_{ij} \otimes E_{ji}, \qquad \Omega^- = \Omega^0 + \sum_{i<j} E_{ji} \otimes E_{ij}.$$

Let $Y = Y_1 \otimes \cdots \otimes Y_M$ be the tensor product of finite-dimensional irreducible $\mathfrak{gl}_N$-modules.

*The Gaudin KZ Hamiltonians* $H^{\mathfrak{G}}_a(\boldsymbol{\lambda}, z)$, $a = 1, \dots, M$, acting on $Y$-valued functions of $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, $z = (z_1, \dots, z_M)$ are defined by the formula:

$$H^{\mathfrak{G}}_a(\boldsymbol{\lambda}, z) = \sum_{i=1}^{N}\left(\lambda_i - \frac{E_{ii}}{2}\right)(E_{ii})^{(a)} + \sum_{b=1, b\neq a}^{M} \frac{z_a(\Omega^+)^{(ab)} + z_b(\Omega^-)^{(ab)}}{z_a - z_b}.$$

Here the linear operator $(\Omega^{\pm})^{(ab)} : Y \to Y$ acts as $\Omega^{\pm}$ on $Y_a \otimes Y_b$, and as the identity on other tensor factors of $Y$. Similarly, $E^{(a)}_{ii}$ acts as $E_{ii}$ on $Y_a$ and as the identity on other factors.

### 8.2. The Gaudin dynamical Hamiltonians

For any $i, j = 1, \dots, N$, $i \neq j$, introduce a series $B_{ij}(t)$ depending on a complex number $t$:

$$B_{i,j}(t) = 1 + \sum_{s=1}^{\infty} (E_{ji})^s (E_{ij})^s \prod_{l=1}^{s} \frac{1}{j(t - E_{ii} + E_{jj} - l)}.$$

The series has a well-defined action in any finite-dimensional $\mathfrak{gl}_N$-module $W$ giving an $\mathrm{End}(W)$-valued rational function of $t$.

*The Gaudin dynamical Hamiltonians* $G_i^{\mathfrak{G}}(\boldsymbol{\lambda}, z)$, $i = 1, \ldots, N$, acting on $Y$-valued functions of $\boldsymbol{\lambda}, z$ are defined by the formula [21]:

$$G_i^{\mathfrak{G}}(\boldsymbol{\lambda}, z) = \bigl(B_{i,N}(\lambda_i - \lambda_N) \ldots B_{i,i+1}(\lambda_i - \lambda_{i+1})\bigr)^{-1} \times \prod_{a=1}^{M} \bigl(z_a^{-E_{ii}}\bigr)^{(a)} \times B_{1,i}(\lambda_1 - \lambda_i) \ldots B_{i-1,i}(\lambda_{i-1} - \lambda_i).$$

*8.3. The Gaudin diagonalization problem*

The Gaudin KZ and dynamical Hamiltonians commute [3,21],

$$\bigl[H_a^{\mathfrak{G}}(\boldsymbol{\lambda}, z), H_b^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\bigr] = 0, \qquad \bigl[H_a^{\mathfrak{G}}(\boldsymbol{\lambda}, z), G_i^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\bigr] = 0, \qquad \bigl[G_i^{\mathfrak{G}}(\boldsymbol{\lambda}, z), G_j^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\bigr] = 0,$$

for $a, b = 1, \ldots, M$, and $i, j = 1, \ldots, N$.

*The Gaudin diagonalization problem* is to diagonalize simultaneously the Gaudin KZ Hamiltonians $H_a^{\mathfrak{G}}$, $a = 1, \ldots, M$, and dynamical Hamiltonians $G_i^{\mathfrak{G}}$, $i = 1, \ldots, N$, for given $\boldsymbol{\lambda}, z$. The Hamiltonians preserve the weight decomposition of $Y$ and the diagonalization problem can be considered on a given weight subspace of $Y$.

*8.4. Diagonalization and critical points*

For a nonnegative integer $m$, denote by $L_m^{\langle N\rangle}$ the irreducible finite-dimensional $\mathfrak{gl}_N$-module with highest weight $(m, 0, \ldots, 0)$.

Let $\boldsymbol{n} = (n_1, \ldots, n_N)$ and $\boldsymbol{m} = (m_1, \ldots, m_M)$ be vectors of nonnegative integers with $\sum_{i=1}^N n_i = \sum_{a=1}^M m_a$.

Consider the tensor product $L_{m_1}^{\langle N\rangle} \otimes \cdots \otimes L_{m_M}^{\langle N\rangle}$ and its weight subspace

$$\boldsymbol{L_m}[\boldsymbol{n}] = \bigl(L_{m_1}^{\langle N\rangle} \otimes \cdots \otimes L_{m_M}^{\langle N\rangle}\bigr)[n_1, \ldots, n_N].$$

Let $\boldsymbol{\lambda} \in \mathbb{C}^N$, $z \in \mathbb{C}^M$. Assume that each of $\boldsymbol{\lambda}$ and $z$ has distinct coordinates. Consider the Hamiltonians $H_a^{\mathfrak{G}}(\boldsymbol{\lambda}, z)$, $G_i^{\mathfrak{G}}(\boldsymbol{\lambda}, z)$ acting on $\boldsymbol{L_m}[\boldsymbol{n}]$. *The Bethe ansatz method* is a method to construct common eigenvectors of the Hamiltonians.

As in Section 6.1 consider the space $\mathbb{C}^{\bar n_1+\cdots+\bar n_{N-1}}$ with coordinates $\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}$. Let $\Phi(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}; \boldsymbol{\lambda}; \boldsymbol{m}; z)$ be the master function on $\mathbb{C}^{\bar n_1+\cdots+\bar n_{N-1}}$ defined in (6.1).

In Section 4 of [9], a certain $\boldsymbol{L_m}[\boldsymbol{n}]$-valued rational function $\omega^{\mathfrak{G}} : \mathbb{C}^{\bar n_1+\cdots+\bar n_{N-1}} \to \boldsymbol{L_m}[\boldsymbol{n}]$, depending on $z$, is constructed. It is called *the Gaudin universal rational function*. For $N = 2$ formulas for the Gaudin universal rational function see below in Section 10.2.

The Gaudin universal rational function is well defined for admissible $\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}$. The Gaudin universal rational function is symmetric with respect to the $\boldsymbol{\Sigma}_{\bar{\boldsymbol{n}}}$-action.

**Theorem 8.1.** *(See [18,9].) If $\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}$ is a Gaudin admissible nondegenerate critical point of the master function $\Phi(\cdot; \boldsymbol{\lambda} - \boldsymbol{n}; \boldsymbol{m}; z)$, then $\omega^{\mathfrak{G}}(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z) \in \boldsymbol{L_m}[\boldsymbol{n}]$ is an eigenvector of the Gaudin Hamiltonians,*

$$H_a^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big) = z_a \frac{\partial}{\partial z_a} \log \Phi\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}; \boldsymbol{\lambda} - \boldsymbol{n}; \boldsymbol{m}; z\big)\omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big), \quad a = 1, \dots, M,$$

$$G_1^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big) = \prod_{a=1}^{M} z_a^{-m_a} \prod_{j=1}^{\bar{n}_1} t_j^{(i)} \omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big),$$

$$G_i^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big) = \prod_{j=1}^{\bar{n}_{i-1}} \big(t_j^{(i-1)}\big)^{-1} \prod_{j=1}^{\bar{n}_i} t_j^{(1)} \omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big), \quad i = 2, \dots, N-1.$$

$$G_N^{\mathfrak{G}}(\boldsymbol{\lambda}, z)\omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big) = \prod_{j=1}^{\bar{n}_{N-1}} \big(t_j^{(N-1)}\big)^{-1} \omega^{\mathfrak{G}}\big(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z\big). \tag{8.1}$$

If $\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}$ is a Gaudin admissible critical point of the master function, then the vector $\omega^{\mathfrak{G}}(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, z)$ is called *a Gaudin Bethe eigenvector*.

By Corollary 7.4, for generic $\boldsymbol{\lambda}$, the number of Gaudin Bethe eigenvectors is not greater than the dimension of the space $\boldsymbol{L}_{\boldsymbol{m}}[\boldsymbol{n}]$. The Bethe ansatz conjecture says that all eigenvectors of the Gaudin Hamiltonians are the Gaudin Bethe eigenvectors in an appropriate sense, cf. [14].

### 8.5. The XXX KZ Hamiltonians

Let $E_{ab}$, $a, b = 1, \dots, M$, be the standard generators of the complex Lie algebra $\mathfrak{gl}_M$.

Let $V$, $W$ be irreducible finite-dimensional $\mathfrak{gl}_M$-modules with highest weight vectors $v \in V$, $w \in W$. The associated rational $R$-matrix is a rational $\mathrm{End}(V \otimes W)$-valued function $R_{VW}(t)$ of a complex variable $t$ uniquely determined by the $\mathfrak{gl}_M$-invariance condition,

$$\big[R_{VW}(t), g \otimes 1 + 1 \otimes g\big] = 0 \quad \text{for any } g \in \mathfrak{gl}_M,$$

the commutation relations

$$R_{VW}(t)\left(t E_{ab} \otimes 1 + \sum_{c=1}^{M} E_{ac} \otimes E_{cb}\right) = \left(t E_{ij} \otimes 1 + \sum_{c=1}^{M} E_{cb} \otimes E_{ac}\right) R_{VW}(t),$$

for any $a, b$, and the normalization condition

$$R_{VW}(t) v \otimes w = v \otimes w.$$

Let $Y = Y_1 \otimes \dots \otimes Y_N$ be the tensor product of finite-dimensional irreducible $\mathfrak{gl}_M$-modules.

*The XXX KZ Hamiltonians* $H_i^{\mathfrak{X}}(z, \boldsymbol{\lambda})$, $i = 1, \dots, N$, acting on $Y$-valued functions of $z = (z_1, \dots, z_M)$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ are defined by the formula:

$$H_i^{\mathfrak{X}}(z, \boldsymbol{\lambda}) = \big(R_{i,N}(\lambda_i - \lambda_N) \dots R_{i,i+1}(\lambda_i - \lambda_{i+1})\big)^{-1} \times \prod_{a=1}^{M} \big(z_a^{-E_{aa}}\big)^{(i)} \times R_{1,i}(\lambda_1 - \lambda_i) \dots R_{i-1,i}(\lambda_{i-1} - \lambda_i),$$

where $R_{jk}(t)$ is the endomorphisms of $Y$ which acts as $R_{Y_j Y_k}(t)$ on $Y_j \otimes Y_k$ and as the identity on other factors of $Y$.

*8.6. The XXX dynamical Hamiltonians*

*The XXX dynamical Hamiltonians* $G_a^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda})$, $a=1,\dots,M$, acting on $Y$-valued functions of $\boldsymbol{z},\boldsymbol{\lambda}$ are defined by the formula [22]:

$$G_a^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda}) = -\frac{E_{aa}^2}{2} + \sum_{i=1}^{N} \lambda_i E_{aa}^{(i)} \\ + \sum_{b=1,b\neq a}^{M} \frac{z_b}{z_a - z_b}(E_{ab}E_{ba} - E_{aa}) + \sum_{b=1}^{M}\sum_{i<j} (E_{ab})^{(i)}(E_{ba})^{(j)}.$$

*8.7. The XXX diagonalization problem*

The XXX KZ and dynamical Hamiltonians commute [4,23],

$$\big[H_i^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda}), H_j^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda})\big]=0, \qquad \big[H_i^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda}), G_a^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda})\big]=0, \qquad \big[G_a^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda}), G_b^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda})\big]=0,$$

for $i,j=1,\dots,N$, and $a,b=1,\dots,M$.

*The XXX diagonalization problem* is to diagonalize simultaneously the XXX Hamiltonians for given $\boldsymbol{z},\boldsymbol{\lambda}$. The Hamiltonians preserve the weight decomposition of $Y$ and the diagonalization problem can be considered on a given weight subspace of $Y$.

*8.8. Diagonalization and solutions of the Bethe ansatz equations*

For a nonnegative integer $n$, denote by $L_n^{\langle M\rangle}$ the irreducible $\mathfrak{gl}_M$-module with highest weight $(n,0,\dots,0)$.

Let $\boldsymbol{m}=(m_1,\dots,m_M)$ and $\boldsymbol{n}=(n_1,\dots,n_N)$ be vectors of nonnegative integers with $\sum_{a=1}^M m_a = \sum_{i=1}^N n_i$.

Consider the tensor product $L_{n_1}^{\langle M\rangle}\otimes\cdots\otimes L_{n_N}^{\langle M\rangle}$ and its weight subspace

$$\boldsymbol{L_n}[\boldsymbol{m}] = \big(L_{n_1}^{\langle M\rangle}\otimes\cdots\otimes L_{n_N}^{\langle M\rangle}\big)[m_1,\dots,m_M].$$

Let $\boldsymbol{z}\in\mathbb{C}^M$, $\boldsymbol{\lambda}\in\mathbb{C}^N$. Assume that each of $\boldsymbol{z}$ and $\boldsymbol{\lambda}$ has distinct coordinates. Consider the Hamiltonians $H_i^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda})$, $G_a^{\mathfrak{X}}(\boldsymbol{z},\boldsymbol{\lambda})$ acting on $\boldsymbol{L_n}[\boldsymbol{m}]$. *The Bethe ansatz method* is a method to construct common eigenvectors of the Hamiltonians.

As in Section 6.2 consider the space $\mathbb{C}^{\bar m_1+\cdots+\bar m_{M-1}}$ with coordinates $\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}$ and the XXX Bethe ansatz equations (6.3). Let $\boldsymbol{1}=(1,\dots,1)$ and

$$\xi^{\langle \boldsymbol{m}\rangle} = \big(\xi_1^{(1)},\dots,\xi_{\bar m_1}^{(1)},\xi_1^{(2)},\dots,\xi_{\bar m_2}^{(2)},\dots,\xi_1^{(M-1)},\dots,\xi_{\bar m_{M-1}}^{(M-1)}\big) \tag{8.2}$$

be the point with coordinates $\xi_a^{(i)}=i$, for $i=1,\dots,M-1$ and $a=1,\dots,\bar m_{M-1}$.

In [19], a certain $\boldsymbol{L_n}[\boldsymbol{m}]$-valued rational function $\omega^{\mathfrak{X}}:\mathbb{C}^{\bar m_1+\cdots+\bar m_{M-1}}\to \boldsymbol{L_n}[\boldsymbol{m}]$, depending on $\boldsymbol{\lambda}$ , is constructed. It is called *the XXX universal rational function*. For $N=2$ formulas for the XXX universal rational function see below in Section 10.3.

The XXX universal rational function $\omega^{\mathfrak{X}}(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda})$ is well defined if $\boldsymbol{t}^{\langle \boldsymbol{m}\rangle} + \xi^{\langle \boldsymbol{m}\rangle}$ is XXX admissible of $(\boldsymbol{\lambda} - \boldsymbol{n} + \boldsymbol{1})$-type. The XXX universal rational function is symmetric with respect to the $\boldsymbol{\Sigma}_{\bar{\boldsymbol{m}}}$-action.

**Theorem 8.2.** *(See [20,12].) If $\boldsymbol{t}^{\langle \boldsymbol{m}\rangle} + \xi^{\langle \boldsymbol{m}\rangle}$ is an XXX admissible solution of $(\boldsymbol{\lambda} - \boldsymbol{n} + \boldsymbol{1})$-type of the Bethe ansatz equations* (6.3)*, then $\omega^{\mathfrak{X}}(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}) \in \boldsymbol{L}_{\boldsymbol{n}}[\boldsymbol{m}]$ is an eigenvector of the XXX Hamiltonians,*

$$H_i^{\mathfrak{X}}(z, \boldsymbol{\lambda})\omega^{\mathfrak{X}}\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big) = \prod_{j=1}^{\bar{m}_1} \frac{t_j^{(1)} - \lambda_i}{t_j^{(1)} - \lambda_i + n_i}\omega^{\mathfrak{X}}\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big), \quad i = 1, \dots, N,$$

$$G_a^{\mathfrak{X}}(z, \boldsymbol{\lambda})\omega^{\mathfrak{X}}\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big) = d_a\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big)\omega^{\mathfrak{X}}\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big), \quad a = 1, \dots, M, \tag{8.3}$$

*where*

$$d_1\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big) = \sum_{i=1}^{N} n_i\left(\lambda_i - \frac{n_i}{2}\right) + \sum_{j=1}^{\bar{m}_1} t_j^{(1)} - \frac{\bar{m}_1}{2} - \sum_{b=2}^{M} \frac{m_b z_b}{z_1 - z_b},$$

$$d_a\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big) = \sum_{j=1}^{\bar{m}_a} t_j^{(a)} - \sum_{j=1}^{\bar{m}_{a-1}} t_j^{(a-1)} - \frac{\bar{m}_{a-1} + \bar{m}_a}{2} - \sum_{b=1}^{a-1} \frac{m_a z_b}{z_a - z_b} - \sum_{b=a+1}^{M} \frac{m_b z_b}{z_a - z_b},$$

$$a = 2, \dots, M-1,$$

$$d_M\big(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}\big) = -\sum_{j=1}^{\bar{m}_{M-1}} t_j^{(a-1)} - \frac{\bar{m}_{M-1}}{2} - \sum_{b=1}^{M-1} \frac{m_M z_b}{z_a - z_b}.$$

If $\boldsymbol{t}^{\langle \boldsymbol{m}\rangle} + \xi^{\langle \boldsymbol{m}\rangle}$ is an XXX admissible solution of $(\boldsymbol{\lambda} - \boldsymbol{n} + \boldsymbol{1})$-type of the XXX Bethe ansatz equations, then the vector $\omega^{\mathfrak{X}}(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda})$ is called *an XXX Bethe eigenvector*.

By Corollary 7.7, for generic $\boldsymbol{z}$ the number of the XXX Bethe eigenvectors is not greater than the dimension of the space $\boldsymbol{L}_{\boldsymbol{n}}[\boldsymbol{m}]$. The Bethe ansatz conjecture says that all eigenvectors of the XXX Hamiltonians are the XXX Bethe eigenvectors.

According to previous discussions, eigenvectors of Gaudin Hamiltonians on $\boldsymbol{L}_{\boldsymbol{m}}[\boldsymbol{n}]$ are related to critical points of the master function $\Phi(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}; \boldsymbol{\lambda} - \boldsymbol{n}; \boldsymbol{m}; z)$. The critical points of $\Phi(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}; \boldsymbol{\lambda} - \boldsymbol{n}; \boldsymbol{m}; z)$ are related to spaces of $(\boldsymbol{n}, \boldsymbol{\lambda} - \boldsymbol{n}, \boldsymbol{m}, z)$-type. The spaces of $(\boldsymbol{n}, \boldsymbol{\lambda} - \boldsymbol{n}, \boldsymbol{m}, z)$-type are rigged bispectral dual to spaces of $(\boldsymbol{m}, z, \boldsymbol{n}, \boldsymbol{\lambda} - \boldsymbol{n} + \boldsymbol{1})$-type, where $\boldsymbol{1} = (1, \dots, 1) \in \mathbb{C}^N$. The spaces of $(\boldsymbol{m}, z, \boldsymbol{n}, \boldsymbol{\lambda} - \boldsymbol{n} + \boldsymbol{1})$-type are related to solutions of the XXX Bethe ansatz equations, which in its turn are related to eigenvectors of the XXX Hamiltonians acting on $\boldsymbol{L}_{\boldsymbol{n}}[\boldsymbol{m}]$. As a result of this chain of relations, the eigenvectors of the Gaudin Hamiltonians on $\boldsymbol{L}_{\boldsymbol{m}}[\boldsymbol{n}]$ and the eigenvectors of the XXX Hamiltonians on $\boldsymbol{L}_{\boldsymbol{n}}[\boldsymbol{m}]$ must be related.

Indeed this relation is given by the $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality.

## 9. The $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality for KZ and dynamical Hamiltonians

### 9.1. The $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality

The Lie algebra $\mathfrak{gl}_N$ acts on $\mathbb{C}[x_1, \dots, x_N]$ by differential operators $E_{ij} \mapsto x_i \frac{\partial}{\partial x_j}$. Denote this $\mathfrak{gl}_N$-module by $L_\bullet$. Then

$$L_\bullet = \bigoplus_{m=0}^{\infty} L_m^{\langle N\rangle},$$

the submodule $L_m^{\langle N\rangle}$ being spanned by homogeneous polynomials of degree $m$. The $\mathfrak{gl}_N$-module $L_m^{\langle N\rangle}$ is irreducible with highest weight $(m, 0, \dots, 0)$ and highest weight vector $x_1^m$.

We consider $\mathfrak{gl}_N$ and $\mathfrak{gl}_M$ simultaneously. To distinguish generators, modules, etc., we indicated the dependence on $N$ and $M$ explicitly, for example, $E_{ij}^{\langle N\rangle}$, $L_m^{\langle M\rangle}$.

Let $P_{MN} = \mathbb{C}[x_{11}, \dots, x_{M1}, \dots, x_{1N}, \dots, x_{MN}]$ be the space of polynomials of $MN$ variables. We define the $\mathfrak{gl}_M$-action on $P_{MN}$ by $E_{ab}^{\langle M\rangle} \mapsto \sum_{i=1}^N x_{ai} \frac{\partial}{\partial x_{bi}}$ and the $\mathfrak{gl}_N$-action by $E_{ij}^{\langle N\rangle} \mapsto \sum_{a=1}^M x_{ai} \frac{\partial}{\partial x_{aj}}$. There are two isomorphisms of vector spaces,

$$\begin{aligned}
&\big(\mathbb{C}[x_1, \dots, x_M]\big)^{\otimes N} \to P_{MN}, \quad (p_1 \otimes \dots \otimes p_N)(x_{11}, \dots, x_{MN}) \mapsto \prod_{i=1}^N p_i(x_{1i}, \dots, x_{Mi}),\\
&\big(\mathbb{C}[x_1, \dots, x_N]\big)^{\otimes M} \to P_{MN}, \quad (p_1 \otimes \dots \otimes p_M)(x_{11}, \dots, x_{MN}) \mapsto \prod_{a=1}^M p_a(x_{a1}, \dots, x_{aN}).
\end{aligned} \tag{9.1}$$

Under these isomorphisms, $P_{MN}$ is isomorphic to $(L_\bullet^{\langle M\rangle})^{\otimes N}$ as a $\mathfrak{gl}_M$-module and to $(L_\bullet{}^{\langle N\rangle})^{\otimes M}$ as a $\mathfrak{gl}_N$-module.

Fix $\boldsymbol{n} = (n_1, \dots, n_N) \in \mathbb{Z}_{\geqslant 0}^N$ and $\boldsymbol{m} = (m_1, \dots, m_M) \in \mathbb{Z}_{\geqslant 0}^M$ with $\sum_{i=1}^N n_i = \sum_{a=1}^M m_a$. Isomorphisms (9.1) induce an isomorphism of the weight subspaces,

$$\boldsymbol{L_n}[\boldsymbol{m}] \simeq \boldsymbol{L_m}[\boldsymbol{n}]. \tag{9.2}$$

Under this isomorphism the KZ and dynamical Hamiltonians interchange,

$$H_a^{\mathfrak{G}, \langle N\rangle}(\boldsymbol{\lambda}, z) = G_a^{\mathfrak{X}, \langle M\rangle}(z, \boldsymbol{\lambda}), \qquad G_i^{\mathrm{G}, \langle N\rangle}(\boldsymbol{\lambda}, z) = H_i^{\mathrm{G}, \langle N\rangle}(z, \boldsymbol{\lambda}),$$

for $a = 1, \dots, M$, $i = 1, \dots, N$, [22].

Recall that $\mathbf{1} = (1, \dots, 1) \in \mathbb{C}^N$, and $\xi^{\langle \boldsymbol{m}\rangle}$ is defined by (8.2).

Let $\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}$ be a Gaudin admissible critical point of the master function $\Phi(\cdot; \boldsymbol{\lambda} - \boldsymbol{n}; \boldsymbol{m}; z)$. Let $(V, z)$ be the associated space of $(\boldsymbol{n}, \boldsymbol{\lambda} - \boldsymbol{n}, \boldsymbol{m}, z)$-type. Let $(U, \boldsymbol{\lambda} - \boldsymbol{n} + \mathbf{1})$ be its rigged bispectral dual space of $(\boldsymbol{m}, z, \boldsymbol{n}, \boldsymbol{\lambda} - \boldsymbol{n} + \mathbf{1})$-type. Assume that $(U, \boldsymbol{\lambda} - \boldsymbol{n} + \mathbf{1})$ is XXX admissible. Let $\boldsymbol{t}^{\langle \boldsymbol{m}\rangle} + \xi^{\langle \boldsymbol{m}\rangle}$ be the associated XXX admissible solution of the Bethe ansatz equations (6.3).

**Conjecture 9.1.** *The corresponding Bethe vectors* $\omega^{\mathfrak{G}}(\boldsymbol{t}^{\langle \boldsymbol{n}\rangle}, \boldsymbol{z}) \in \boldsymbol{L_m}[\boldsymbol{n}]$ *and* $\omega^{\mathfrak{X}}(\boldsymbol{t}^{\langle \boldsymbol{m}\rangle}, \boldsymbol{\lambda}) \in \boldsymbol{L_n}[\boldsymbol{m}]$ *are proportional under the duality isomorphism* (9.2).

We prove this conjecture for $N = M = 2$ in Section 10.4, see also [13].

## 10. The case of $N = M = 2$

### *10.1. The* $(\mathfrak{gl}_2, \mathfrak{gl}_2)$ *duality*

Let $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{m} = (m_1, m_2)$ be two vectors of nonnegative integers such that $n_1 + n_2 = m_1 + m_2$.

Fix $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{z} = (z_1, z_2)$ each with distinct coordinates.

For $m \in \mathbb{Z}_{\geqslant 0}$, let $L_m$ be the irreducible $\mathfrak{gl}_2$-module with highest weight $(m, 0)$ and highest weight vector $v_m$. The vectors $E_{21}^i v_m$, $i = 0, \dots, m$, form a basis in $L_m$.

Set

$$\alpha = \max(0, n_2 - m_1), \qquad \beta = \min(m_2, n_2).$$

The vectors

$$\frac{E_{21}^{n_2-i} v_{m_1}}{(n_2-i)!} \otimes \frac{E_{21}^i v_{m_2}}{i!}, \quad \alpha \leqslant i \leqslant \beta,$$

form a basis in $\boldsymbol{L_m}[\boldsymbol{n}] = (L_{m_1} \otimes L_{m_2})[n_1, n_2]$. The vectors

$$\frac{E_{21}^{m_2-i} v_{n_1}}{(m_2-i)!} \otimes \frac{E_{21}^i v_{n_2}}{i!}, \quad \alpha \leqslant i \leqslant \beta,$$

form a basis in $\boldsymbol{L_n}[\boldsymbol{m}] = (L_{n_1} \otimes L_{n_2})[m_1, m_2]$. Isomorphism (9.2) identifies the vectors with the same index $i$.

We consider the Gaudin Hamiltonians on $\boldsymbol{L_m}[\boldsymbol{n}]$ and the XXX Hamiltonians on $\boldsymbol{L_n}[\boldsymbol{m}]$.

Let $B_{i,j}(t)$ be the series defined in Section 8.2. Consider the map

$$b(\boldsymbol{\lambda}) : (L_{m_1} \otimes L_{m_2})[n_1, n_2] \to (L_{m_1} \otimes L_{m_2})[n_2, n_1] \tag{10.1}$$

defined by the formula

$$\frac{E_{21}^{n_2-i} v_{m_1}}{(n_2-i)!} \otimes \frac{E_{21}^i v_{m_2}}{i!} \mapsto B_{21}(\lambda_1 - \lambda_2) \frac{E_{21}^{m_1-n_2+i} v_{m_1}}{(m_1-n_2+i)!} \otimes \frac{E_{21}^{m_2-i} v_{m_2}}{(m_2-i)!}.$$

For a generic $\boldsymbol{\lambda}$ this map is an isomorphism.

By [21], the map $b(\boldsymbol{\lambda})$ commutes with the Gaudin Hamiltonians in the following sense:

$$b(\boldsymbol{\lambda}) H_a^{\mathfrak{G}}(\lambda_1, \lambda_2, \boldsymbol{z}) = H_a^{\mathfrak{G}}(\lambda_2, \lambda_1, \boldsymbol{z}) b(\boldsymbol{\lambda}),$$
$$b(\boldsymbol{\lambda}) G_i^{\mathfrak{G}}(\lambda_1, \lambda_2, \boldsymbol{z}) = G_i^{\mathfrak{G}}(\lambda_2, \lambda_1, \boldsymbol{z}) b(\boldsymbol{\lambda})$$

for $a, i = 1, 2$.

The map

$$\frac{E_{21}^{m_2-i} v_{n_1}}{(m_2-i)!} \otimes \frac{E_{21}^{i} v_{n_2}}{i!} \mapsto \frac{E_{21}^{n_1-m_2+i} v_{n_1}}{(n_1-m_2+i)!} \otimes \frac{E_{21}^{n_2-i} v_{n_2}}{(n_2-i)!} \tag{10.2}$$

defines the Weyl isomorphism

$$s : (L_{n_1} \otimes L_{n_2})[m_1, m_2] \to (L_{n_1} \otimes L_{n_2})[m_2, m_1]. \tag{10.3}$$

The Weyl isomorphism commutes with the XXX Hamiltonians,

$$s H_i^{\mathfrak{X}}(\boldsymbol{z}, \boldsymbol{\lambda}) = H_i^{\mathfrak{X}}(\boldsymbol{z}, \boldsymbol{\lambda}) s,$$
$$s G_a^{\mathfrak{X}}(\boldsymbol{z}, \boldsymbol{\lambda}) = G_a^{\mathfrak{X}}(\boldsymbol{z}, \boldsymbol{\lambda}) s$$

for $i, a = 1, 2$.

*10.2. The Gaudin universal rational function and the fundamental differential equation*

Let $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{m} = (m_1, m_2)$ be two vectors of nonnegative integers such that $n_1 + n_2 = m_1 + m_2$.

Fix $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{z} = (z_1, z_2)$ each with distinct coordinates. *The Gaudin universal rational $\boldsymbol{L_m}[\boldsymbol{n}]$-valued function* is the function

$$\omega(t_1, \ldots, t_{n_2}) = \sum_i C_i \frac{E_{21}^{n_2-i} v_{m_1}}{(n_2-i)!} \otimes \frac{E_{21}^{i} v_{m_2}}{i!},$$

where

$$C_i(t_1, \ldots, t_{n_2}) = \operatorname{Sym}_{n_2}\left[\prod_{j=1}^{n_2-i} \frac{1}{t_j - z_1} \prod_{j=1}^{i} \frac{1}{t_{n_2+j-i} - z_2}\right] \tag{10.4}$$

and $\operatorname{Sym}_n f(t_1, \ldots, t_n) = \sum_{\sigma \in \Sigma_n} f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$, see [9].

The Gaudin universal rational function is symmetric with respect to the group $\Sigma_{n_2}$ of permutations of variables $t_1, \ldots, t_{n_2}$. The $\Sigma_{n_2}$-orbit of a point $\boldsymbol{t}^{\langle \boldsymbol{n} \rangle} = (t_1, \ldots, t_{n_2})$ is represented by the polynomial $p(x) = (x - t_1) \cdots (x - t_{n_2})$.

The polynomials $(x - z_1)^{n_2 - i}(x - z_2)^i$, $i = 0, \ldots, n_2$, form a basis in the space of polynomials in $x$ of degree not greater than $n_2$.

**Lemma 10.1.** *Let $p(x) = (x - t_1) \cdots (x - t_{n_2})$ be a polynomial. Let the numbers $C_0, \ldots, C_{n_2}$ be given by formula* (10.4). *Then*

$$\frac{(z_1 - z_2)^{n_2} p(x)}{p(z_1) p(z_2)} = \sum_{i=0}^{n_2} (-1)^i C_i \frac{(x - z_1)^{n_2 - i}}{(n_2 - i)!} \frac{(x - z_2)^i}{i!}.$$

The proof is straightforward.

Let $V = \langle x^{\lambda_1} p_1(x), x^{\lambda_2} p_2(x)\rangle$. Let $(V, \boldsymbol{z})$ be a space of $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, \boldsymbol{z})$-type. The rigged fundamental differential operator of $(V, \boldsymbol{z})$ has the form

$$\begin{aligned} D = {} & (x - z_1)(x - z_2)(x\partial_x - \lambda_1 - n_1)(x\partial_x - \lambda_2) \\ & + \phi_{11} x(x - z_1)(x\partial_x - \lambda_1 - n_1) + \phi_{12} z_2 (x - z_1)(x\partial_x - \lambda_2) \\ & + \phi_{21} x(x - z_2)(x\partial_x - \lambda_1 - n_1) + \phi_{22} z_1 (x - z_2)(x\partial_x - \lambda_2), \end{aligned}$$

where $\phi_{ij}$ are suitable numbers such that

$$\begin{aligned} \phi_{21} + \phi_{22} &= -m_1, & \phi_{11} + \phi_{12} &= -m_2, \\ \phi_{22} + \phi_{12} &= -n_1, & \phi_{21} + \phi_{11} &= -n_2. \end{aligned} \tag{10.5}$$

The operator $D$ can be written also in the form

$$\begin{aligned} D = {} & (x - z_1)(x - z_2)(x\partial_x - \lambda_1 - n_1)(x\partial_x - \lambda_2 - n_2) \\ & + \psi_{11} x(x - z_1)(x\partial_x - \lambda_1) + \psi_{12} z_2 (x - z_1)(x\partial_x - \lambda_2 - n_2) \\ & + \psi_{21} x(x - z_2)(x\partial_x - \lambda_1) + \psi_{22} z_1 (x - z_2)(x\partial_x - \lambda_2 - n_2), \end{aligned}$$

where $\psi_{ij}$ are such that

$$\begin{aligned} \psi_{21} + \psi_{22} &= -m_1, & \psi_{11} + \psi_{12} &= -m_2, \\ \psi_{22} + \psi_{12} &= -n_1, & \psi_{21} + \psi_{11} &= -n_2, \end{aligned}$$

and $\psi_{11}(\lambda_1 - \lambda_2 - n_2) = \phi_{11}(\lambda_1 - \lambda_2 + n_1) + m_2 n_2$.

Equation $Df = 0$ has a solution

$$f(x) = x^{\lambda_2} p_2(x) = x^{\lambda_2} \sum_{i=0}^{n_2} c_i \frac{(x - z_1)^{n_2 - i}}{(n_2 - i)!} \frac{(x - z_2)^i}{i!}.$$

Due to conditions (10.5), the expression $x^{-\lambda_2} Df(x)$ is a polynomial of degree at most $n_2 + 1$ vanishing at $x = 0$. Expanding this polynomial as a linear combination of the polynomials $x(x - z_1)^{n_2 - i}(x - z_2)^i$, $i = 0, \dots, n_2$, we obtain that the equation $Df = 0$ is equivalent to the following relations for the coefficients $c_0, \dots, c_{n_2}$:

$$\begin{aligned} & z_1 i(m_1 - n_2 + i)c_{i-1} + z_2(m_2 - i)(n_2 - i)c_{i+1} \\ & \quad + \bigl((z_1 + z_2)i^2 + \bigl(z_2 m_1 - z_1 m_2 - 2 z_2 n_2 - (\lambda_1 - \lambda_2)(z_1 - z_2)\bigr)i \\ & \quad - (\lambda_1 - \lambda_2 + n_1)(z_1 - z_2)\phi_{11} + z_2 m_2 n_2\bigr)c_i = 0, \end{aligned} \tag{10.6}$$

$i = 0, \dots, n_2$. Notice that values $i = 0, n_2 - m_1, m_2, n_2$ are the values of $i$ for which Eq. (10.6) does not contain $c_{i-1}$ or $c_{i+1}$.

**Lemma 10.2.** *Eqs.* (10.6) *with $i$ such that $\alpha \leqslant i \leqslant \beta$ form a closed system of equations with respect to $c_j$ such that $\alpha \leqslant j \leqslant \beta$.*

The proof is straightforward.

Eqs. (10.6) have a symmetry. Namely, (10.6) does not change if we replace the parameters $\lambda_1, \lambda_2, n_1, n_2, \phi_{11}, i$ by $\lambda_2, \lambda_1, n_2, n_1, \psi_{12}, m_2 - i$, respectively, and after that replace the unknowns $c_j$ by $(z_1/z_2)^j c_{m_2-j}$.

*10.3. The XXX universal rational function and the fundamental difference equation*

Let $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{m} = (m_1, m_2)$ be two vectors of nonnegative integers such that $n_1 + n_2 = m_1 + m_2$.

Fix $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{z} = (z_1, z_2)$ each with distinct coordinates. *The XXX universal rational $\boldsymbol{L_n}[\boldsymbol{m}]$-valued function* is the function

$$\omega(s_1, \dots, s_{m_2}, \boldsymbol{\lambda}) = \sum_i C_i \frac{E_{21}^{m_2-i} v_{n_1}}{(m_2 - i)!} \otimes \frac{E_{21}^i v_{n_2}}{i!},$$

where

$$C_i(s_1, \dots, s_{m_2}, \boldsymbol{\lambda}) = \mathrm{Sym}_{m_2}\left[\prod_{a=1}^{m_2-i} \frac{1}{s_a - \lambda_1 + n_1} \prod_{b=m_2-i+1}^{m_2} \frac{s_b - \lambda_1}{(s_b - \lambda_1 + n_1)(s_b - \lambda_2 + n_2)} \right.$$
$$\left. \times \prod_{a=1}^{m_2-i} \prod_{b=m_2-i+1}^{m_2} \frac{s_a - s_b - 1}{s_a - s_b}\right], \tag{10.7}$$

see [8,20].

The XXX universal rational function is symmetric with respect to the group $\Sigma_{m_2}$ of permutations of variables $s_1, \dots, s_{m_2}$. The $\Sigma_{m_2}$-orbit of a point $s^{\langle \boldsymbol{n} \rangle} = (s_1, \dots, s_{m_2})$ is represented by the polynomial $q(u) = (u - s_1) \cdots (u - s_{m_2})$.

The polynomials $\prod_{j=0}^{m_2-i-1}(u - \lambda_1 + n_1 - j) \prod_{j=0}^{i-1}(u - \lambda_2 + j)$, $i = 0, \dots, m_2$, form a basis in the space of polynomials in $x$ of degree not greater than $m_2$.

**Lemma 10.3.** *Let $q(u) = (u - s_1) \cdots (u - s_{m_2})$ be a polynomial. Let the numbers $C_0, \dots, C_{m_2}$ be given by formula* (10.7). *Then*

$$\frac{q(u)}{q(\lambda_1 - n_1) q(\lambda_2 - n_2)} \prod_{j=0}^{m_2-1} (\lambda_1 - \lambda_2 + n_2 - j)$$
$$= \sum_{i=0}^{m_2} (-1)^i \frac{C_i(s_1, \dots, s_{m_2}, \boldsymbol{\lambda})}{(m_2 - i)! i!} \prod_{j=0}^{m_2-i-1} (u - \lambda_1 + j) \prod_{j=0}^{i-1} (u - \lambda_2 + n_2 - j).$$

The proof is straightforward.

Let $U = \langle z_1^u q_1(u), z_2^u q_2(u)\rangle$. Let $(U, \boldsymbol{\lambda})$ be a space of $(\boldsymbol{m}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{\lambda})$-type. The rigged fundamental difference operator of $(U, \boldsymbol{\lambda})$ has the form

$$\begin{aligned}\widehat{D} &= (u-\lambda_1-n_1+1)(u-\lambda_2+1)(\tau-z_1)(\tau-z_2)\\ &\quad+\varphi_{11}(u-\lambda_1-n_1+1)\tau(\tau-z_1)+\varphi_{12}(u-\lambda_2+1)z_2(\tau-z_1)\\ &\quad+\varphi_{21}(u-\lambda_1-n_1+1)\tau(\tau-z_2)+\varphi_{22}(u-\lambda_2+1)z_1(\tau-z_2),\end{aligned}$$

where $\varphi_{ij}$ are suitable numbers such that

$$\begin{aligned}\varphi_{21}+\varphi_{22}&=-m_1, &\qquad \varphi_{11}+\varphi_{12}&=-m_2,\\ \varphi_{22}+\varphi_{12}&=-n_1, &\qquad \varphi_{21}+\varphi_{11}&=-n_2.\end{aligned}\tag{10.8}$$

Equation $\widehat{D}f = 0$ has a solution

$$f = z_2^u q_2(u) = z_2^u \sum_{i=1}^{m_2} \frac{c_i}{(m_2-i)!i!} \prod_{j=0}^{m_2-i-1} (u-\lambda_1-n_1+j) \prod_{j=0}^{i-1} (u-\lambda_2-j).$$

Due to conditions (10.8), the expression $z_2^{-u}\widehat{D}f(u)$ is a polynomial of degree at most $m_2$. Expanding this polynomial as a linear combination of the polynomials

$$\prod_{j=0}^{m_2-i-1} (u-\lambda_1-n_1+j+1) \prod_{j=0}^{i-1} (u-\lambda_2-j+1), \quad i = 0, \dots, m_2,$$

we obtain that the equation $\widehat{D}f = 0$ is equivalent to the following relations for the coefficients $c_0, \dots, c_{m_2}$:

$$\begin{aligned}&z_1 i(m_1-n_2+i)c_{i-1} + z_2(m_2-i)(n_2-i)c_{i+1}\\ &\quad+\big((z_1+z_2)i^2+\big(z_2m_1-z_1m_2-2z_2n_2-(\lambda_1-\lambda_2)(z_1-z_2)\big)i\\ &\quad-(\lambda_1-\lambda_2+n_1)(z_1-z_2)\varphi_{11}+z_2m_2n_2\big)c_i = 0,\end{aligned}\tag{10.9}$$

$i = 0, \dots, m_2$. Notice that values $i = 0, n_2 - m_1, m_2, n_2$ are the values of $i$ for which Eq. (10.6) does not contain $c_{i-1}$ or $c_{i+1}$.

**Lemma 10.4.** *Eqs.* (10.9) *with $i$ such that $\alpha \leqslant i \leqslant \beta$ form a closed system of equations with respect to $c_j$ such that $\alpha \leqslant j \leqslant \beta$.*

The proof is straightforward.

Eqs. (10.9) have a symmetry. Namely, (10.9) does not change if we replace the parameters $z_1, z_2, m_1, m_2, \varphi_{11}, i$ by $z_2, z_1, m_2, m_1, \varphi_{21}, n_2 - i$, respectively, and after that replace the unknowns $c_j$ by $c_{n_2-j}$. Another symmetry of (10.9) is that the equations do not change if we replace $\lambda_1, \lambda_2$ by $\lambda_1 + 1, \lambda_2 + 1$.

The pair of Eqs. (10.6) and (10.9) have a symmetry. Namely, Eq. (10.6) turns into Eq. (10.9) if we replace the parameter $\phi_{11}$ there by $\varphi_{11}$.

*10.4. Proof of Conjecture 9.1 for $N = M = 2$*

The symmetries of Eqs. (10.6) and (10.9) imply the following theorem.

**Theorem 10.5.** *Let $V = \langle x^{\lambda_1} p_1(x), x^{la_2} p_2(x)\rangle$ and $U = \langle z_1^u q_1(u), z_2^u q_2(u)\rangle$. Let $(V, z)$ be a space of $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{m}, z)$-type and $(U, \boldsymbol{\lambda} + \mathbf{1})$ the rigged bispectral dual space of $(\boldsymbol{m}, z, \boldsymbol{n}, \boldsymbol{\lambda} + \mathbf{1})$-type, where $\mathbf{1} = (1, 1)$. Write*

$$p_2(x) = \sum_{i=0}^{n_2} c_i(\boldsymbol{\lambda}) \frac{(x - z_1)^{n_2 - i}}{(n_2 - i)!} \frac{(x - z_2)^i}{i!},$$

$$p_1(x) = \sum_{i=0}^{n_1} d_i(\boldsymbol{\lambda}) \frac{(x - z_1)^{n_1 - i}}{(n_1 - i)!} \frac{(x - z_2)^i}{i!},$$

$$q_2(u) = \sum_{i=1}^{m_2} \frac{e_i(\boldsymbol{\lambda} + \mathbf{1})}{(m_2 - i)!i!} \prod_{j=0}^{m_2 - i - 1} (u - \lambda_1 - 1 - n_1 + j) \prod_{j=0}^{i-1} (u - \lambda_2 - 1 - j),$$

$$q_1(u) = \sum_{i=1}^{m_1} \frac{f_i(\boldsymbol{\lambda} + \mathbf{1})}{(m_1 - i)!i!} \prod_{j=0}^{m_1 - i - 1} (u - \lambda_1 - 1 - n_1 + j) \prod_{j=0}^{i-1} (u - \lambda_2 - 1 - j).$$

*Then there exist nonzero numbers $A(\boldsymbol{\lambda}), B(\boldsymbol{\lambda}), C(\boldsymbol{\lambda})$ such that*

$$c_i(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda}) d_{m_2 - i}(\boldsymbol{\lambda}) (z_1/z_2)^i = B(\boldsymbol{\lambda}) e_i(\boldsymbol{\lambda} + \mathbf{1}) = C(\boldsymbol{\lambda}) f_{n_2 - i}(\boldsymbol{\lambda} + \mathbf{1})$$

*for all $i$, $\alpha \leqslant i \leqslant \beta$.*

Theorems 10.5 and 8.1 yield the next statement.

**Corollary 10.6.** *The Bethe vectors*

$$\sum_{i=\alpha}^{\beta} c_i(\boldsymbol{\lambda} + \boldsymbol{n}) \frac{E_{21}^{n_2 - i} v_{m_1}}{(n_2 - i)!} \otimes \frac{E_{21}^i v_{m_2}}{i!}, \qquad \sum_{i=\alpha}^{\beta} d_i(\boldsymbol{\lambda} + \boldsymbol{n}) \frac{E_{21}^{n_1 - i} v_{m_1}}{(n_1 - i)!} \otimes \frac{E_{21}^i v_{m_2}}{i!},$$

$$\sum_{i=\alpha}^{\beta} e_i(\boldsymbol{\lambda} + \boldsymbol{n}) \frac{E_{21}^{m_2 - i} v_{n_1}}{(m_2 - i)!} \otimes \frac{E_{21}^i v_{n_2}}{i!}, \qquad \sum_{i=\alpha}^{\beta} f_i(\boldsymbol{\lambda} + \boldsymbol{n}) \frac{E_{21}^{m_1 - i} v_{n_1}}{(m_1 - i)!} \otimes \frac{E_{21}^i v_{n_2}}{i!}$$

*are identified up to proportionality by isomorphisms* (10.1) *and* (9.2).

The fact that the first and second Bethe vectors are identified by isomorphism (10.1) is proved also in [16] in a different way.

*10.5. Correspondence of solutions of the Bethe ansatz equations for $N = M = 2$*

Consider the following four systems of the Bethe ansatz equations:

$$\frac{\lambda_1 - \lambda_2 - 1}{t_i} + \sum_{a=1}^{2} \frac{m_a}{t_i - z_a} - \sum_{j=1, j \neq i}^{n_2} \frac{2}{t_i - t_j} = 0, \quad i = 1, \ldots, n_2, \tag{10.10}$$

$$\frac{\lambda_1 - \lambda_2 - 1}{t_i} + \sum_{a=1}^{2} \frac{m_a}{t_i - z_a} - \sum_{j=1, j \neq i}^{n_1} \frac{2}{t_i - t_j} = 0, \quad i = 1, \ldots, n_1, \tag{10.11}$$

$$\prod_{i=1}^{2} \frac{t_a - \lambda_i - 1}{t_a - \lambda_i - 1 - n_i} \prod_{b, b \neq a}^{m_2} \frac{t_a - t_b - 1}{t_b - t_b + 1} = \frac{z_2}{z_1}, \quad a = 1, \ldots, m_2, \tag{10.12}$$

$$\prod_{i=1}^{2} \frac{t_a - \lambda_i - 1}{t_a - \lambda_i - 1 - n_i} \prod_{b, b \neq a}^{m_1} \frac{t_a - t_b - 1}{t_b - t_b + 1} = \frac{z_2}{z_1}, \quad a = 1, \ldots, m_1. \tag{10.13}$$

The first two systems are of the Gaudin type and the last two systems are of the XXX type, see (6.2) and (6.3).

Set $d = \dim(L_{m_1} \otimes L_{m_2})[n_1, n_2] = \dim(L_{m_1} \otimes L_{m_2})[n_2, n_1] = \dim(L_{n_1} \otimes L_{n_2})[m_1, m_2] = \dim(L_{n_1} \otimes L_{n_2})[m_2, m_1]$. By [17] for generic $\boldsymbol{\lambda}$ and $\boldsymbol{z}$ each of the first two systems has exactly $d$ orbits of Gaudin admissible solutions. By [17] for generic $\boldsymbol{\lambda}$ and $\boldsymbol{z}$ each of the last two systems has exactly $d$ orbits of XXX admissible solutions.

**Theorem 10.7.** *Let $\boldsymbol{\lambda}$ and $\boldsymbol{z}$ be generic. Let $(t_1, \ldots, t_{n_2})$ be a Gaudin admissible solution of the system* (10.10). *Let $(V, \boldsymbol{z}) = \langle x^{\lambda_1} p_1(x), x^{\lambda_2} p_2(x) \rangle$ be the space of $(\boldsymbol{\lambda}, \boldsymbol{z}, \boldsymbol{n}, \boldsymbol{m})$-type associated with the orbit of the critical point. Let $(U, \boldsymbol{\lambda} + \mathbf{1}) = \langle z_1^u q_1(u), z_2^u q_2(u) \rangle$, where $\mathbf{1} = (1, 1)$ , be the rigged bispectral dual space to $(V, \boldsymbol{z})$. Then the polynomial $p_1$ represents the orbit of a Gaudin admissible solution of the system* (10.11)*, the polynomial $q_2$ represents the orbit of an XXX admissible solution of the system* (10.12)*, and the polynomial $q_1$ represents the orbit of an XXX admissible solution of the system* (10.13).

## 11. Baker–Akhiezer functions and bispectral correspondence

In this section we indicate a connection of our integral transforms of Section 4 with the bispectral theory, see [24,7,6,2,1].

*11.1. Grassmannian of Gaudin admissible nondegenerate spaces*

For $\lambda \in \mathbb{C}$, we call a vector subspace $X_\lambda \subset \mathbb{C}[u]$ *Gaudin admissible at* $\lambda$ if there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that

$$(u - \lambda)(u - \lambda - 1) \cdots (u - \lambda - m + 1)\mathbb{C}[u] \subset X_\lambda$$

and there exists $f \in X_\lambda$ such that $f(\lambda) \neq 0$. We call a vector subspace $X \subset \mathbb{C}[u]$ *Gaudin admissible* if

$$X = \bigcap_{i=1}^{n} X_{\lambda_i}$$

where $n$ is a natural number, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ if $i \neq j$, and for any $i$, $X_{\lambda_i}$ is Gaudin admissible at $\lambda_i$.

Each $X_{\lambda_i}$ can be defined by a finite set of linear equations. Namely, there exist a positive integer $N_i$, a sequence of integers $0 < n_{i1} < \dots < n_{iN_i}$ and complex numbers $c_{i,j,a}$, $a = 0, \dots, n_{ij}$, such that $c_{i,j,n_{ij}} \neq 0$ for all $i, j$. Then the space $X_{\lambda_i}$ consists of all polynomials $r \in \mathbb{C}[u]$ satisfying the equations

$$\sum_{a=0}^{n_{ij}} c_{i,j,a} r(\lambda_i + a) = 0 \quad \text{for } j = 1, \dots, N_i. \tag{11.1}$$

For a Gaudin admissible subspace $X$, define the complex vector space $V$ as the space spanned by functions $x^{\lambda_i} p_{ij}(x)$, $i = 1, \dots, n$, $j = 1, \dots, N_i$, where $p_{ij}(x) = \sum_{a=0}^{n_{ij}} c_{i,j,a} x^a$. The space $V$ is a space of quasi-polynomials of the type considered in Section 2.1.

On the other hand, having a space $V$ of quasi-polynomials, as in Section 2.1, we can recover a Gaudin admissible subspace $X \subset \mathbb{C}[u]$ by formula (11.1).

We say that the space $X$ is *nondegenerate* if the corresponding space $V$ is nondegenerate in the sense of Section 2.1.

We denote the set of all Gaudin admissible nondegenerate subspaces by $\mathrm{Gr}^{\mathfrak{G}}$ and call it *the Grassmannian* of the Gaudin admissible nondegenerate subspaces.

Let $X \subset \mathbb{C}[u]$ be a Gaudin admissible subspace and $V$ the associated space of quasi-polynomials in $x$. Define the algebra $A_X = \{p \in \mathbb{C}[u] \mid p(u)X \subset X\}$. An equivalent definition is $A_X = \{p \in \mathbb{C}[u] \mid p(x\partial_x)V \subset V\}$.

Let $\bar{D}_V$ be the monic fundamental differential operator of $V$. The function

$$\Psi_X(x, u) = \prod_{i=1}^{n} \prod_{j=1}^{N_i} \frac{x}{u - \lambda_i - n_{ij}} \bar{D}_V x^u$$

will be called *the stationary Baker–Akhiezer function of the Gaudin admissible space* $X$. Introduce the rational function $\psi_X(x, u)$ by the formula $\Psi_X(x, u) = \psi_X(x, u) x^u$. The function $\psi_X(x, u)$ expands as a power series in $x^{-1}, u^{-1}$ of the form

$$\psi_X(x, u) = 1 + \sum_{i,j=1}^{\infty} c_{ij} x^{-i} u^{-j}.$$

It is easy to see that for every $p \in A_X$, there exists a linear differential operator $L_p(x, \partial_x)$ with rational coefficients such that $L_p(x, \partial_x)\bar{D}_V = \bar{D}_V p(x\partial_x)$. As a corollary, we conclude that

$$L_p(x, \partial_x)\Psi_X(x, u) = p(u)\Psi_X(x, u).$$

For $p_1, p_2 \in A$, the corresponding operators $L_{p_1}(x, \partial_x)$ and $L_{p_2}(x, \partial_x)$ commute.

*11.2. Grassmannian of XXX admissible nondegenerate spaces*

Let $z \in \mathbb{C}^*$ be a nonzero complex number with fixed argument. We call a vector subspace $Y_z \subset \mathbb{C}[x]$ *XXX admissible at $z$* if there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $(x-z)^n\mathbb{C}[x] \subset Y_z$ and there exists $f \in Y_z$ such that $f(z) \neq 0$. We call a vector subspace $Y \subset \mathbb{C}[x]$ *XXX admissible* if

$$Y = \bigcap_{a=1}^{m} Y_{z_a}$$

where $m$ is a natural number, $z_1, \dots, z_m \in \mathbb{C}^*$ are distinct numbers and for any $a$, the space $Y_{z_a}$ is XXX admissible at $z_a$.

Each $Y_{z_a}$ can be defined by a finite set of linear equations. Namely, there exist a positive integer $M_a$, a sequence of integers $0 < m_{a1} < \cdots < m_{aM_a}$ and complex numbers $d_{a,b,i}$, $i = 0, \dots, m_{ab}$, such that $d_{a,b,m_{ab}} \neq 0$ for all $a, b$. Then the space $Y_{z_a}$ consists of all polynomials $r \in \mathbb{C}[x]$ satisfying the equations

$$\sum_{i=0}^{m_{ab}} d_{a,b,i}(x\partial_x)^i r(z_a) = 0 \quad \text{for } b = 1, \dots, M_a. \tag{11.2}$$

For an XXX admissible subspace $Y$ define the complex vector space $U$ as the space spanned by functions $z_a^u q_{ab}(u)$, $a = 1, \dots, m$, $b = 1, \dots, M_a$, where

$$z_a^u q_{ab}(u) = z_a^u \sum_{i=0}^{m_{ab}} d_{a,b,i} u^i.$$

The space $U$ is a space of quasi-exponentials in $u$ of the type considered in Section 3.1.

On the other hand, having a space $U$ of quasi-exponentials, as in Section 3.1, we can recover an XXX admissible subspace $Y \subset \mathbb{C}[x]$ by formula (11.2).

We say that $Y$ is *nondegenerate* if the corresponding $U$ is nondegenerate in the sense of Section 3.6.

We denote the set of all XXX admissible nondegenerate subspaces by $\mathrm{Gr}^{\mathfrak{X}}$ and call it *the Grassmannian* of the XXX admissible nondegenerate subspaces.

Let $Y \subset \mathbb{C}[x]$ be an XXX admissible subspace and $U$ the associated space of quasi-exponentials. Define the algebra $A_Y = \{p \in \mathbb{C}[x] \mid p(x)W \subset W\}$. An equivalent definition is $A_Y = \{p \in \mathbb{C}[x] \mid p(\tau_u)U \subset U\}$.

Let $\bar{D}_U$ be the monic fundamental difference operator of $U$. The function

$$\Phi_Y(u, x) = \prod_{a=1}^{m} (x - z_a)^{-Ma} \bar{D}_U x^u$$

will be called *the stationary Baker–Akhiezer function of the XXX admissible space $Y$*. Introduce the rational function $\phi_Y(u, x)$ by the formula $\Phi_Y(u, x) = \phi_Y(u, x)x^u$. The function $\phi_Y(u, x)$ expands as a power series in $u^{-1}, x^{-1}$ of the form

$$\phi_Y(u, x) = 1 + \sum_{i,j=1}^{\infty} c_{ij} u^{-i} x^{-j}.$$

It is easy to see that for every $p \in A_Y$, there exists a linear difference operator $L_p(u,\tau_u)$ with rational coefficients such that $L_p(u,\tau_u)\bar{D}_V = \bar{D}_V\, p(\tau_u)$. As a corollary, we conclude that

$$L_p(u,\tau_u)\Phi_Y(u,x) = p(x)\Phi_Y(u,x).$$

For $p_1, p_2 \in A$, the corresponding operators $L_{p_1}(u,\tau_u)$ and $L_{p_2}(u,\tau_u)$ commute.

*11.3. Bispectral correspondence*

In terms of the Baker–Akhiezer functions, we introduce a *bispectral* correspondence between points of the Grassmannian of Gaudin admissible nondegenerate subspaces and points of the Grassmannian of XXX admissible nondegenerate subspaces. We say that a Gaudin admissible nondegenerate space $X$ corresponds to an XXX admissible nondegenerate space $Y$ if $\Psi_X(x,u) = \Phi_Y(u,x)$. This correspondence is an analog of Wilson's bispectral correspondence in [24], see also the papers [7,6,2,1].

**Theorem 11.1.** *Let $V$ be a nondegenerate space of quasi-polynomials and let $U$ be the nondegenerate space of quasi-exponentials which are bispectral dual with respect to the integral transforms of Section* 4. *Then the corresponding admissible spaces $X \in \mathrm{Gr}^{\mathfrak{G}}$ and $Y \in \mathrm{Gr}^{\mathfrak{X}}$ are bispectral correspondent.*

## Acknowledgments

Authors thank Yu. Berest and O. Chalykh for discussions and for providing the authors with manuscripts [2,1] after the present work had been written.

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